Problem 5

In each of Problems 1 through 8, solve the given differential equation.

$$y' = (\cos^2 x)(\cos^2 2y)$$

Solution

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$$

Bring the terms with y to the left and bring the terms with x to the right.

$$\frac{dy}{\cos^2 2y} = \cos^2 x\,dx$$

$$\sec^2 2y\,dy = \cos^2 x\,dx$$

Integrate both sides.

$$\int \sec^2 2y\,dy = \int \cos^2 x\,dx \tag{1}$$

Use the power-reducing formula for cosine on the right and make the following substitution on the left.

$$u = 2y$$

$$du = 2\,dy \quad \to \quad \frac{du}{2} = dy$$

Equation (1) becomes

$$\int \sec^2 u \left(\frac{du}{2}\right) = \int \frac{1}{2}(1 + \cos 2x)\,dx$$

$$\frac{1}{2}\int \sec^2 u\,du = \int \frac{1}{2}\,dx + \int \frac{1}{2}\cos 2x\,dx$$

$$\frac{1}{2}\tan u = \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C$$

$$\frac{1}{2}\tan 2y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Multiply both sides by 2.

$$\tan 2y = x + \frac{1}{2}\sin 2x + 2C$$

Take the inverse tangent of both sides.

$$2y = \tan^{-1}\left(x + \frac{1}{2}\sin 2x + 2C\right)$$

Therefore, using a new constant C_1 for $2C$,

$$y(x) = \frac{1}{2}\tan^{-1}\left(x + \frac{1}{2}\sin 2x + C_1\right).$$