Problem 2

A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution

Let t represent the time in minutes, let $V = V(t)$ represent the volume in liters, and let $m = m(t)$ represent the mass of salt in grams. The tank initially contains 120 L and no salt.

$$V(0) = 120 \text{ L}$$
$$m(0) = 0 \text{ g}$$

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time.

$$\begin{aligned}\frac{dV}{dt} &= 2\ \frac{\text{L}}{\text{min}} - 2\ \frac{\text{L}}{\text{min}} \\ &= 0\end{aligned}$$

Integrate both sides with respect to t.

$$V(t) = C_1$$

Use the initial condition for V to determine C_1.

$$V(0) = C_1 = 120 \text{ L}$$

So the volume is

$$V(t) = 120 \text{ L}.$$

Now apply the law to the mass, noting that dm/dt is the rate that the mass of salt increases with respect to time. To obtain the rate of mass flow, multiply the concentration by the volume flow rate. Since the solution is well-stirred, the concentration flowing out is $m(t)/V(t)$.

$$\begin{aligned}\frac{dm}{dt} &= \left(2\ \frac{\text{L}}{\text{min}}\right)\left(\gamma\ \frac{\text{g}}{\text{L}}\right) - \left(2\ \frac{\text{L}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= 2\gamma - \frac{m}{60}\end{aligned}$$

Bring $m/60$ to the left side.

$$\frac{dm}{dt} + \frac{m}{60} = 2\gamma$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{ds}{60}\right) = e^{t/60}$$

Proceed with the multiplication.

$$e^{t/60}\frac{dm}{dt} + \frac{1}{60}e^{t/60}m = 2\gamma e^{t/60}$$

The left side can be written as $d/dt(Im)$ by the product rule.

$$\frac{d}{dt}(e^{t/60}m) = 2\gamma e^{t/60}$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{t/60}m &= \int^t 2\gamma e^{s/60}\, ds + C_2 \\ &= 120\gamma e^{t/60} + C_2 \end{aligned}$$

Divide both sides by $e^{t/60}$.

$$m(t) = 120\gamma + C_2 e^{-t/60}$$

Use the initial condition for m to determine C_2.

$$m(0) = 120\gamma + C_2 = 0 \quad \rightarrow \quad C_2 = -120\gamma$$

So then the mass of salt is

$$\begin{aligned} m(t) &= 120\gamma - 120\gamma e^{-t/60} \\ &= 120\gamma(1 - e^{-t/60}) \text{ g.} \end{aligned}$$

In the limit as $t \to \infty$, the exponential function tends to zero. Therefore,

$$\lim_{t\to\infty} m(t) = 120\gamma \text{ g.}$$