Problem 5

A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}(1+\frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

(a) Find the amount of salt in the tank at any time.

(b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.

(c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

Solution

Part (a)

Let t represent the time in minutes, let $V = V(t)$ represent the volume in gallons, and let $m = m(t)$ represent the mass of salt in ounces. The tank initially contains 100 gal of water and 50 oz of salt.

$$\begin{aligned} V(0) &= 100 \text{ gal} \\ m(0) &= 50 \text{ oz} \end{aligned}$$

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time.

$$\begin{aligned} \frac{dV}{dt} &= 2\ \frac{\text{gal}}{\text{min}} - 2\ \frac{\text{gal}}{\text{min}} \\ &= 0 \end{aligned}$$

Integrate both sides with respect to t.

$$V(t) = C_1$$

Use the initial condition for V to determine C_1.

$$V(0) = C_1 = 100 \text{ gal}$$

So the volume is

$$V(t) = 100 \text{ gal.}$$

Now apply the law to the mass, noting that dm/dt is the rate that the mass of salt increases with respect to time. To obtain the rate of mass flow, multiply the concentration by the volume flow rate. Assuming the solution is well-stirred, the concentration flowing out is $m(t)/V(t)$.

$$\begin{aligned} \frac{dm}{dt} &= \left(2\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{1}{4}(1+\frac{1}{2}\sin t)\ \frac{\text{oz}}{\text{gal}}\right) - \left(2\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= \frac{1}{2}(1+\frac{1}{2}\sin t) - \frac{m}{50} \end{aligned}$$

Bring $m/50$ to the left side.

$$\frac{dm}{dt} + \frac{m}{50} = \frac{1}{2} + \frac{1}{4}\sin t$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{ds}{50}\right) = e^{t/50}$$

Proceed with the multiplication.

$$e^{t/50}\frac{dm}{dt} + \frac{1}{50}e^{t/50}m = \frac{1}{2}e^{t/50} + \frac{1}{4}e^{t/50}\sin t$$

The left side can be written as $d/dt(Im)$ by the product rule.

$$\frac{d}{dt}(e^{t/50}m) = \frac{1}{2}e^{t/50} + \frac{1}{4}e^{t/50}\sin t$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{t/50}m &= \int^t \left(\frac{1}{2}e^{s/50} + \frac{1}{4}e^{s/50}\sin s\right) ds + C_2 \\ &= \int^t \frac{1}{2}e^{s/50}\,ds + \int^t \frac{1}{4}e^{s/50}\sin s\,ds + C_2 \\ &= 25e^{t/50} + \frac{1}{4}\int^t e^{s/50}\sin s\,ds + C_2 \qquad (1) \end{aligned}$$

Use integration by parts twice to determine the remaining integral.

$$\begin{aligned} \int^t e^{s/50}\sin s\,ds &= \int^t e^{s/50}\frac{d}{ds}(-\cos s)\,ds \\ &= e^{s/50}(-\cos s)\Big|^t - \int^t \frac{1}{50}e^{s/50}(-\cos s)\,ds \\ &= -e^{t/50}\cos t + \frac{1}{50}\int^t e^{s/50}\cos s\,ds \\ &= -e^{t/50}\cos t + \frac{1}{50}\int^t e^{s/50}\frac{d}{ds}(\sin s)\,ds \\ &= -e^{t/50}\cos t + \frac{1}{50}\left[e^{s/50}(\sin s)\Big|^t - \int^t \frac{1}{50}e^{s/50}\sin s\,ds\right] \\ &= -e^{t/50}\cos t + \frac{1}{50}e^{t/50}\sin t - \frac{1}{2500}\int^t e^{s/50}\sin s\,ds \end{aligned}$$

Solve this equation for the integral.

$$\frac{2501}{2500}\int^t e^{s/50}\sin s\,ds = \frac{e^{t/50}}{50}(\sin t - 50\cos t)$$

$$\int^t e^{s/50}\sin s\,ds = \frac{50}{2501}e^{t/50}(\sin t - 50\cos t)$$

Substitute this result into equation (1).

$$e^{t/50}m = 25e^{t/50} + \frac{25}{5002}e^{t/50}(\sin t - 50\cos t) + C_2$$

Divide both sides by $e^{t/50}$.

$$m(t) = 25 + \frac{25}{5002}(\sin t - 50\cos t) + C_2e^{-t/50}$$

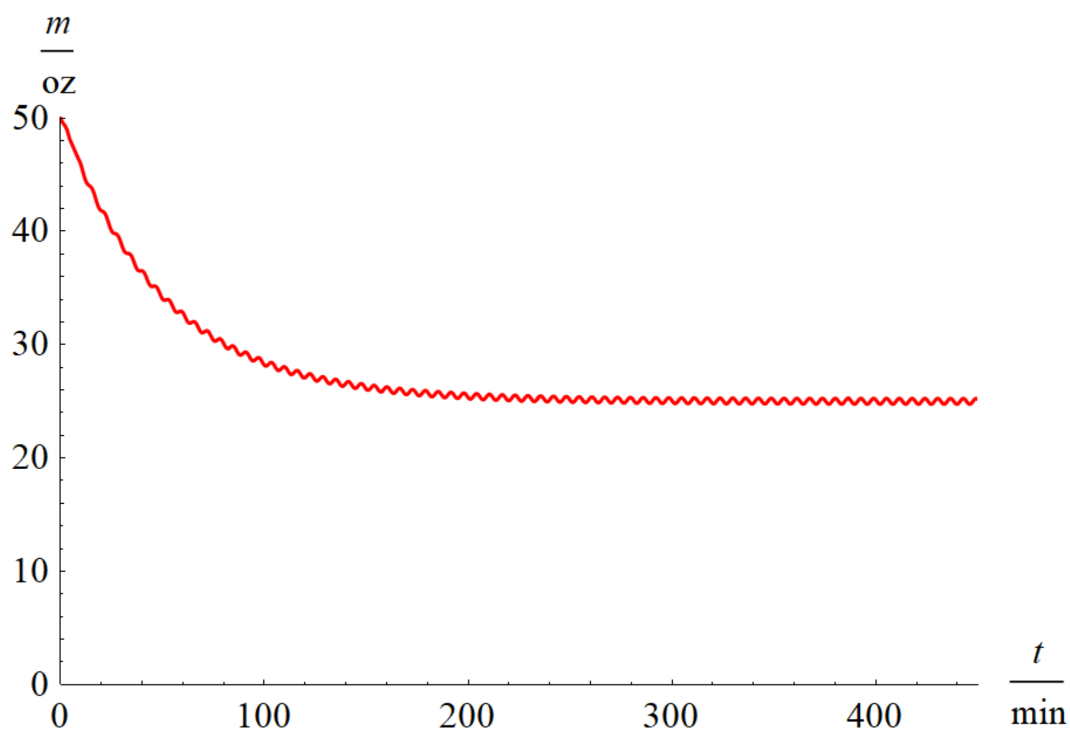
Use the initial condition for m to determine C_2.

$$m(0) = \frac{61\,900}{2501} + C_2 = 50 \quad \rightarrow \quad C_2 = \frac{63\,150}{2501}$$

So then the mass of salt in the tank is

$$m(t) = 25 + \frac{25}{5002}(\sin t - 50\cos t) + \frac{63\,150}{2501}e^{-t/50}.$$

Part (b)



Part (c)

Take the limit of $m(t)$ as $t \to \infty$ to determine the steady state. The term with the exponential function vanishes.

$$\begin{aligned}\lim_{t\to\infty} m(t) &= 25 + \frac{25}{5002}(\sin t - 50\cos t)\\ &= 25 + \frac{25}{5002}\sin t - \frac{625}{2501}\cos t\end{aligned}$$

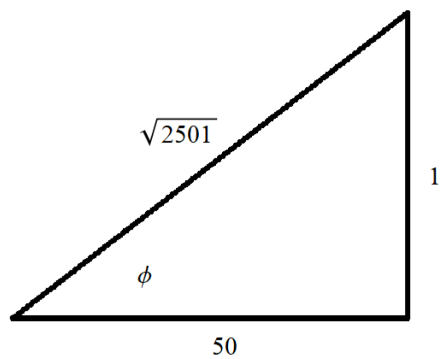
Our aim now is to express the last two terms as a single sinusoidal function $A\cos(\omega t + \phi)$. Let

$$\begin{aligned}A\sin\phi &= \frac{25}{5002}\\ A\cos\phi &= \frac{625}{2501}\end{aligned}$$

and solve this system of two equations for the two unknowns.

$$\tan\phi = \frac{\frac{25}{5002}}{\frac{625}{2501}} = \frac{1}{50} \quad \to \quad \phi = \tan^{-1}\frac{1}{50}$$

Draw the implied right triangle to determine either $\sin\phi$ or $\cos\phi$.



We see that

$$\sin\phi = \frac{1}{\sqrt{2501}},$$

so

$$A\left(\frac{1}{\sqrt{2501}}\right) = \frac{25}{5002} \quad \to \quad A = \frac{25}{5002}\sqrt{2501}.$$

As a result,

$$\begin{aligned}\lim_{t\to\infty} m(t) &= 25 + A\sin\phi\sin t - A\cos\phi\cos t\\ &= 25 - A(\cos t\cos\phi - \sin t\sin\phi)\\ &= 25 - A\cos(t+\phi)\\ &= 25 - \frac{25}{5002}\sqrt{2501}\cos\left(t + \tan^{-1}\frac{1}{50}\right).\end{aligned}$$

Therefore, the amplitude of oscillation is

$$A = \frac{25}{5002}\sqrt{2501} \approx 0.24995,$$

and the constant level about which oscillations take place is 25.