Problem 6

Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's[2] principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h.

(a) Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

(b) By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}, \tag{i}$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. The value of α for water is about 0.6.

(c) Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m, and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

Solution

Part (a)

According to Torricelli's principle, the outflow velocity is the same as the speed an object has if it falls a distance h by gravity. Use one of the kinematic formulas for constant acceleration to determine v (the one without t), orienting the positive y-axis in the direction of gravity.

$$\begin{aligned} v^2 &= v_0^2 + 2a\Delta y \\ &= 0 + 2(g)h \\ &= 2gh \end{aligned}$$

Taking the square root of both sides, we obtain the desired result.

$$v = \sqrt{2gh}$$

Part (b)

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

[2]Evangelista Torricelli (1608–1647), successor to Galileo as court mathematician in Florence, published this result in 1644. He is also known for constructing the first mercury barometer and for making important contributions to geometry.

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time. Note also that multiplying the outflow velocity by the area αa the fluid flows through gives a volume rate of change flowing out.

$$\frac{dV}{dt} = 0 - (\alpha a)v$$

Substitute the result of part (a) for v.

$$\frac{dV}{dt} = -\alpha a\sqrt{2gh}$$

Use the fact that volume is cross-sectional area times thickness: $dV = A(h)\,dh$. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt},$$

so the previous equation becomes

$$\frac{dV}{dh}\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

Therefore,

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

Part (c)

For a right circular cylinder, the cross-sectional area $A(h)$ is constant.

$$A\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

Solve this ODE by separating variables.

$$\frac{dh}{\sqrt{h}} = -\frac{\alpha a\sqrt{2g}}{A}\,dt$$

Integrate both sides.

$$2\sqrt{h} = -\frac{\alpha a\sqrt{2g}}{A}t + C$$

Since the tank is full initially, the initial condition is $h(0) = 3$. Use it to determine C.

$$2\sqrt{3} = C$$

So then

$$2\sqrt{h} = -\frac{\alpha a\sqrt{2g}}{A}t + 2\sqrt{3}.$$

Set $h = 0$ and solve for t to determine the time it takes for the fluid height to drop to the level of the outlet.

$$0 = -\frac{\alpha a\sqrt{2g}}{A}t + 2\sqrt{3}$$

$$\begin{aligned} t &= \frac{2\sqrt{3}A}{\alpha a\sqrt{2g}} \\ &= \frac{2\sqrt{3}(\pi \cdot 1^2)}{0.6 \cdot \pi(0.1)^2\sqrt{2 \cdot 9.81}} \\ &\approx 130.\text{ seconds} \end{aligned}$$