Problem 10

A home buyer can afford to spend no more than \$1500/month on mortgage payments. Suppose that the interest rate is 6%, that interest is compounded continuously, and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow on a 20-year mortgage; on a 30-year mortgage.

(b) Determine the total interest paid during the term of the mortgage in each of the cases in part (a).

Solution

The amount of money $S(t)$ that the buyer has to pay changes in time due to two factors, the compound interest and his continuous payments. The rate of growth for compounding is rS, and the rate of decay due to the continuous payments is k.

$$\frac{dS}{dt} = rS - k$$

Bring rS to the left side.

$$\frac{dS}{dt} - rS = -k$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t (-r)\,ds\right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt}\frac{dS}{dt} - re^{-rt}S = -ke^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt}S) = -ke^{-rt}$$

Integrate both sides with respect to t.

$$e^{-rt}S = \frac{k}{r}e^{-rt} + C$$

Multiply both sides by e^{rt}.

$$S(t) = \frac{k}{r} + Ce^{rt}$$

Apply the initial condition $S(0) = S_0$ to determine C.

$$S(0) = \frac{k}{r} + C = S_0 \quad \rightarrow \quad C = S_0 - \frac{k}{r}$$

Therefore, the amount of money the buyer has to pay after t years is

$$\begin{aligned} S(t) &= \frac{k}{r} + \left(S_0 - \frac{k}{r}\right)e^{rt} \\ &= \frac{k}{r}(1 - e^{rt}) + S_0 e^{rt}. \end{aligned}$$

Set $k = 1500 \times 12 = 18\,000$ dollars/year and $r = 6\% = 0.06$.

$$S(t) = \frac{18\,000}{0.06}(1 - e^{0.06t}) + S_0 e^{0.06t}$$

For a 20-year mortgage, $S(t) = 0$ at $t = 20$.

$$0 = \frac{18\,000}{0.06}(1 - e^{0.06 \cdot 20}) + S_0 e^{0.06 \cdot 20} \quad \rightarrow \quad S_0 \approx \$209\,641.74$$

For a 30-year mortgage, $S(t) = 0$ at $t = 30$.

$$0 = \frac{18\,000}{0.06}(1 - e^{0.06 \cdot 30}) + S_0 e^{0.06 \cdot 30} \quad \rightarrow \quad S_0 \approx \$250\,410.33$$

These values of S_0 are how much the buyer can afford to borrow initially. For the 20-year mortgage, he pays a total of $\$18\,000 \times 20 = \$360\,000$, so the total interest he pays is

$$\$360\,000 - \$209\,641.74 \approx \$150\,358.26.$$

For the 30-year mortgage, he pays a total of $\$18\,000 \times 30 = \$540\,000$, so the total interest he pays is

$$\$540\,000 - \$250\,410.33 \approx \$289\,589.67.$$