Problem 14

Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$dy/dt = (0.5 + \sin t)y/5.$$

(a) If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.

(b) Suppose that the growth rate is replaced by its average value $1/10$. Determine the doubling time τ in this case.

(c) Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ?

(d) Plot the solutions obtained in parts (a), (b), and (c) on a single set of axes.

Solution

Part (a)

The ODE describing the population is separable.

$$\frac{dy}{dt} = \frac{0.5 + \sin t}{5} y$$

Separate variables.

$$\frac{dy}{y} = \frac{0.5 + \sin t}{5}\, dt$$

Integrate both sides.

$$\ln y = 0.1t - 0.2 \cos t + C$$

Exponentiate both sides.

$$\begin{aligned} y(t) &= e^{0.1t - 0.2\cos t + C} \\ &= e^C e^{0.1t - 0.2\cos t} \end{aligned}$$

Use a new constant A for e^C.

$$y(t) = Ae^{0.1t - 0.2\cos t} \tag{1}$$

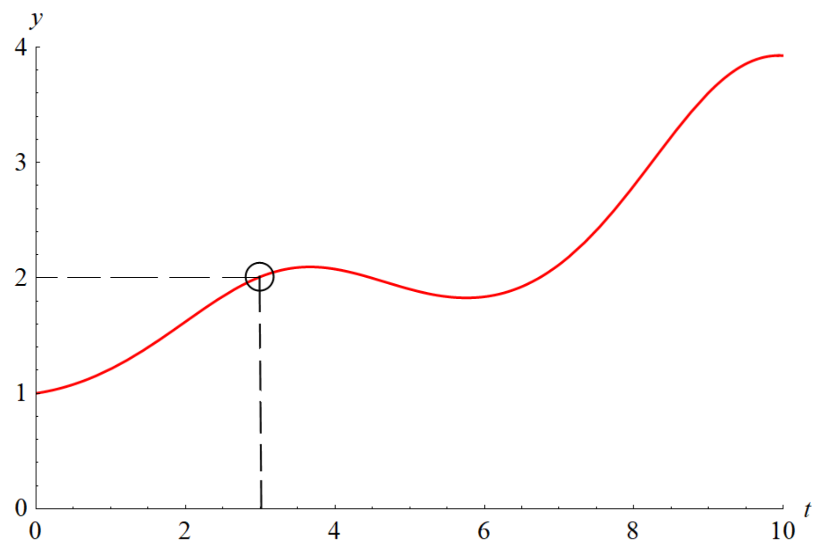
Apply the initial condition $y(0) = 1$ to determine A.

$$y(0) = Ae^{-0.2} = 1 \quad \rightarrow \quad A = e^{0.2}$$

Therefore, the population is

$$\begin{aligned} y(t) &= e^{0.2} e^{0.1t - 0.2\cos t} \\ &= e^{0.1t + 0.2(1 - \cos t)}. \end{aligned}$$

Plot $y(t)$ versus t and find when the initial population doubles to $y = 2$.

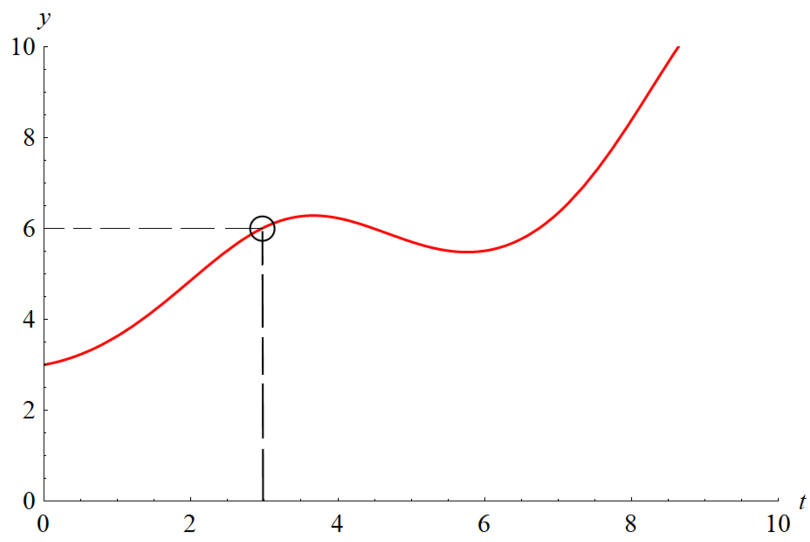


From the graph we see that the population doubles at about $\tau \approx 3$. Rather than $y(0) = 1$, let's apply $y(0) = 3$ to the general solution in equation (1).

$$y(0) = Ae^{-0.2} = 3 \quad \rightarrow \quad A = 3e^{0.2}$$

In this case the population is

$$\begin{aligned} y(t) &= 3e^{0.2}e^{0.1t-0.2\cos t} \\ &= 3e^{0.1t+0.2(1-\cos t)}. \end{aligned}$$



The population still doubles at $\tau \approx 3$.

Part (b)

The average of the function of t in front of y in the ODE over one period is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{0.5+\sin t}{5}\,dt=\frac{1}{10}.$$

Use this value in the ODE instead.

$$\frac{dy}{dt}=\frac{1}{10}y$$

Divide both sides by y.

$$\frac{\frac{dy}{dt}}{y}=\frac{1}{10}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln y=\frac{1}{10}$$

Integrate both sides with respect to t.

$$\ln y=\frac{1}{10}t+C_1$$

Exponentiate both sides.

$$\begin{aligned}y(t)&=e^{t/10+C_1}\\&=e^{C_1}e^{t/10}\end{aligned}$$

Use a new constant A_1 for e^{C_1}.

$$y(t)=A_1e^{t/10}$$

Use the initial condition $y(0)=y_0$ to determine A_1. The point of using a general initial condition $y(0)=y_0$ is to show that the doubling constant is independent of y_0, the initial population.

$$y(0)=A_1=y_0$$

So then

$$y(t)=y_0e^{t/10}.$$

Set $y(t)=2y_0$ and solve for $t=\tau$ to determine the doubling constant.

$$2y_0=y_0e^{\tau/10}$$

$$2=e^{\tau/10}$$

$$\ln 2=\ln e^{\tau/10}$$

$$\ln 2=\frac{\tau}{10}$$

$$\tau=10\ln 2\approx 6.93$$

For $y_0=1$, the population is

$$y(t)=e^{t/10}.$$

Part (c)

The ODE describing the population is still separable.

$$\frac{dy}{dt} = \frac{0.5 + \sin 2\pi t}{5} y$$

Separate variables.

$$\frac{dy}{y} = \frac{0.5 + \sin 2\pi t}{5}\, dt$$

Integrate both sides.

$$\ln y = \frac{1}{10}t - \frac{1}{10\pi}\cos 2\pi t + C_2$$

Exponentiate both sides.

$$\begin{aligned} y(t) &= \exp\left(\frac{1}{10}t - \frac{1}{10\pi}\cos 2\pi t + C_2\right) \\ &= e^{C_2} \exp\left(\frac{1}{10}t - \frac{1}{10\pi}\cos 2\pi t\right) \end{aligned}$$

Use a new constant A_2 for e^{C_2}.

$$y(t) = A_2 \exp\left(\frac{1}{10}t - \frac{1}{10\pi}\cos 2\pi t\right)$$

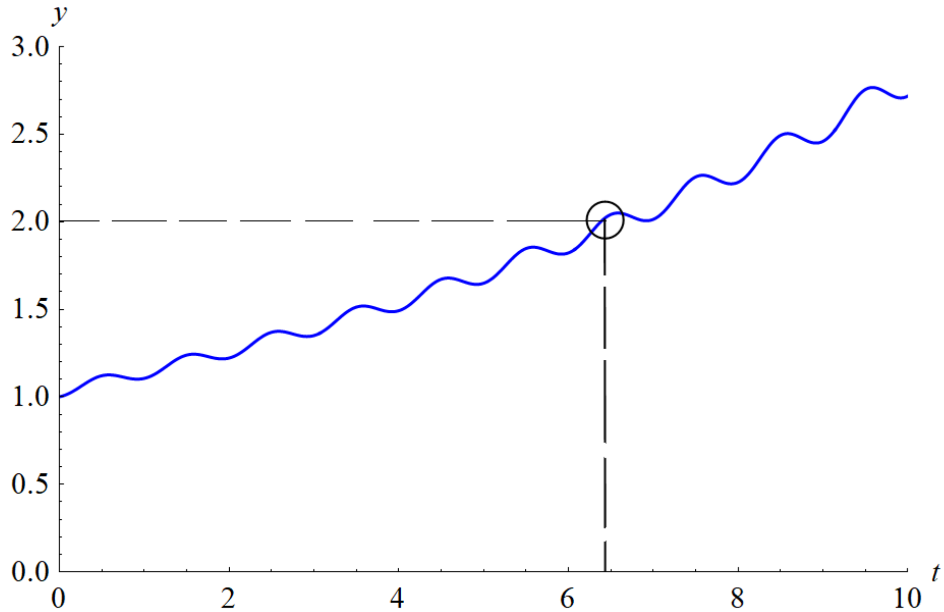
Apply the initial condition $y(0) = 1$ to determine A_2.

$$y(0) = A_2 \exp\left(-\frac{1}{10\pi}\right) = 1 \quad \rightarrow \quad A_2 = \exp\left(\frac{1}{10\pi}\right)$$

Therefore, the population is

$$\begin{aligned} y(t) &= \exp\left(\frac{1}{10\pi}\right) \exp\left(\frac{1}{10}t - \frac{1}{10\pi}\cos 2\pi t\right) \\ &= \exp\left[\frac{1}{10}t + \frac{1}{10\pi}(1 - \cos 2\pi t)\right]. \end{aligned}$$

Plot $y(t)$ versus t to find when the initial population doubles to $y = 2$.



From the graph we see that the population doubles at about $\tau \approx 6.4$.

Part (d)