Problem 15

Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \qquad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$, and k represents the rate of predation.

(a) Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between 1/2 and 1.

(b) Estimate the critical initial population y_c below which the population will become extinct.

(c) Choose other values of k and find the corresponding y_c for each one.

(d) Use the data you have found in parts (b) and (c) to plot y_c versus k.

Solution

Part (a)

The ODE to solve is

$$\frac{dy}{dt} = \frac{1 + \sin t}{5} y - \frac{1}{5}.$$

Bring the term with y to the left side.

$$\frac{dy}{dt} - \frac{1 + \sin t}{5} y = -\frac{1}{5}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t -\frac{1 + \sin s}{5}\, ds\right) = \exp\left(-\frac{1}{5}t + \frac{1}{5}\cos t\right) = e^{(-t+\cos t)/5}$$

Proceed with the multiplication.

$$e^{(-t+\cos t)/5}\frac{dy}{dt} - \frac{1 + \sin t}{5} e^{(-t+\cos t)/5} y = -\frac{1}{5} e^{(-t+\cos t)/5}$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}[e^{(-t+\cos t)/5} y] = -\frac{1}{5} e^{(-t+\cos t)/5}$$

Integrate both sides with respect to t.

$$e^{(-t+\cos t)/5} y = \int^t -\frac{1}{5} e^{(-s+\cos s)/5}\, ds + C$$

The lower limit of integration is arbitrary as long as C is present, so it will be set to zero. Bring the constant in front of the integral.

$$e^{(-t+\cos t)/5} y = -\frac{1}{5}\int_0^t e^{(-s+\cos s)/5}\, ds + C$$

Multiply both sides by $e^{(t-\cos t)/5}$ to solve for y.

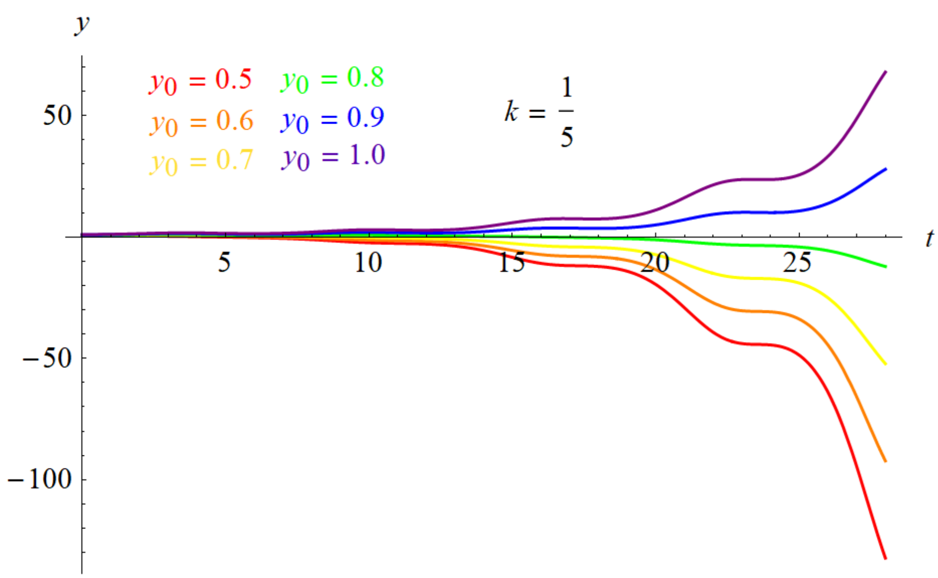
$$y(t) = e^{(t-\cos t)/5}\left[-\frac{1}{5}\int_0^t e^{(-s+\cos s)/5}\,ds + C\right]$$

Apply the initial condition $y(0) = y_0$.

$$y(0) = e^{-1/5}(C) = y_0 \quad \rightarrow \quad C = y_0 e^{1/5}$$

Therefore,

$$y(t) = e^{(t-\cos t)/5}\left[-\frac{1}{5}\int_0^t e^{(-s+\cos s)/5}\,ds + y_0 e^{1/5}\right].$$



Part (b)

The population doesn't drop to zero if y_0 is somewhere between 0.8 and 0.9 (closer to 0.8).

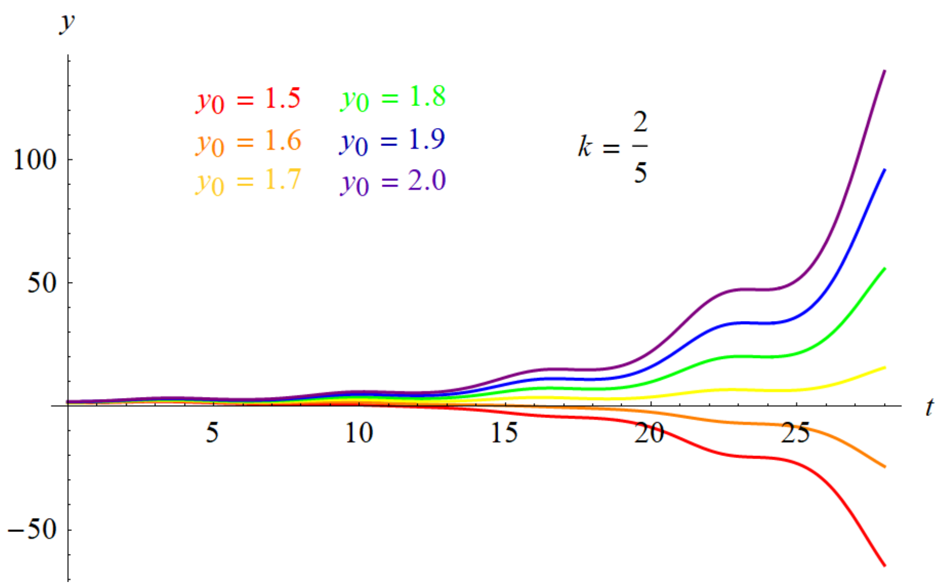
$$y_c \approx 0.82$$

Part (c)

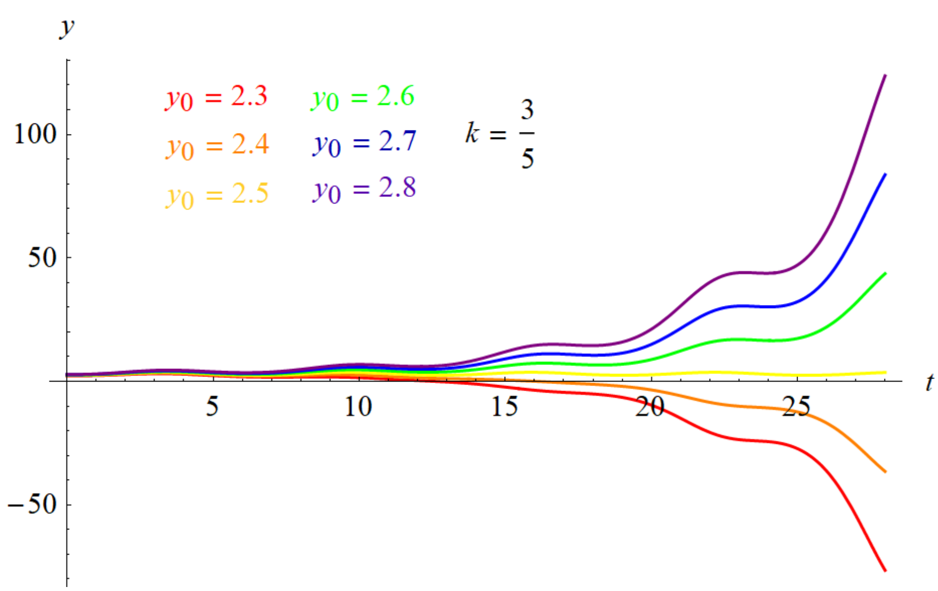
For general k, the solution is

$$y(t) = e^{(t-\cos t)/5}\left[-k\int_0^t e^{(-s+\cos s)/5}\,ds + y_0 e^{1/5}\right].$$

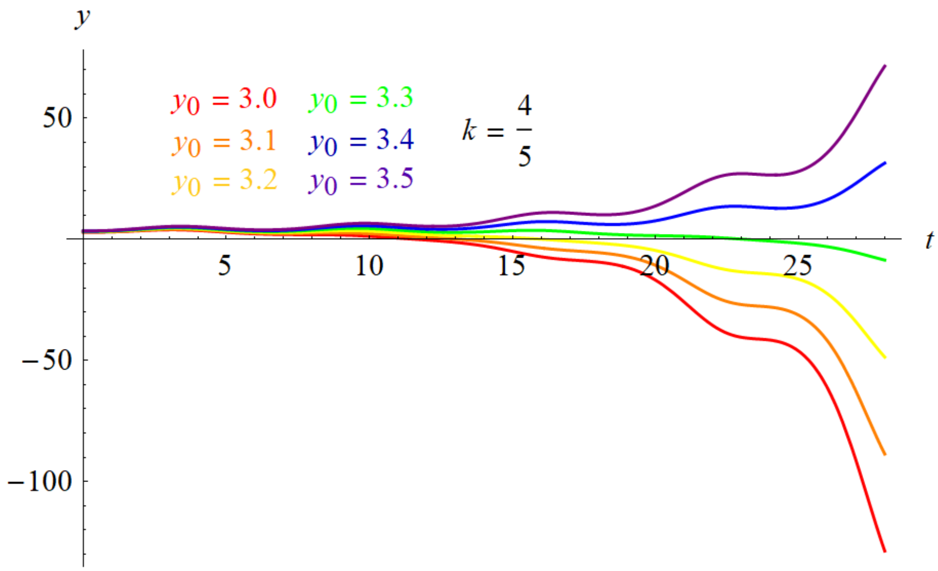
Different values of k will be chosen, and the solution will be plotted for several values of y_0.



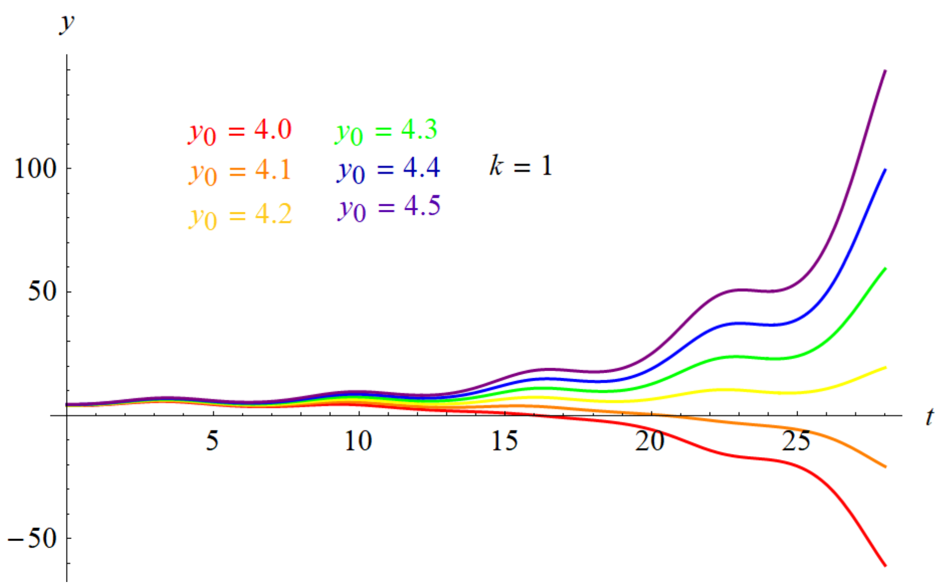
The critical value of the initial population is $y_c \approx 1.67$.



The critical value of the initial population is $y_c \approx 2.50$.



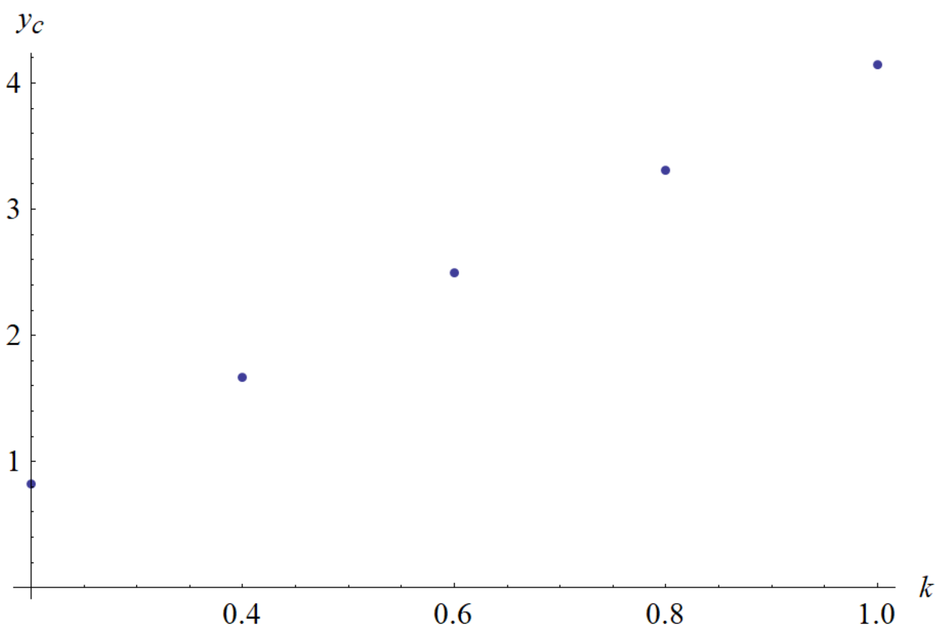
The critical value of the initial population is $y_c \approx 3.31$.



The critical value of the initial population is $y_c \approx 4.15$.

Part (d)

The points (k, y_c) collected in parts (b) and (c) are as follows: $(0.2, 0.82)$, $(0.4, 1.67)$, $(0.6, 2.5)$, $(0.8, 3.31)$, and $(1, 4.15)$.



The graph appears to be linear. Use the first and last points to determine the slope and the line going through them.

$$m \approx \frac{4.15 - 0.82}{1 - 0.2} = 4.1625$$

$$y_c - 4.15 \approx 4.1625(k - 1)$$

$$y_c \approx 4.1625k - 0.0125$$