Problem 16

Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

Solution

Newton's law of cooling can be expressed mathematically as a proportionality. Let $T = T(t)$ be the temperature of the coffee; dT/dt then represents the rate the temperature changes. Also, let T_0 be the temperature of the surroundings.

$$\frac{dT}{dt} \propto -(T - T_0)$$

The minus sign in front of the parentheses accounts for the fact that the temperature of an object will decrease if it's hotter than the environment. Introduce a proportionality constant k on the right to change this to an equation we can use.

$$\frac{dT}{dt} = -k(T - T_0)$$

Solve this ODE by separating variables.

$$\frac{dT}{T - T_0} = -k\,dt$$

Integrate both sides.

$$\ln|T - T_0| = -kt + C$$

Exponentiate both sides.

$$\begin{aligned} |T - T_0| &= e^{-kt+C} \\ &= e^C e^{-kt} \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign.

$$T - T_0 = \pm e^C e^{-kt}$$

Use a new constant A for $\pm e^C$.

$$T - T_0 = Ae^{-kt}$$

Therefore, the temperature of the coffee obeys

$$T(t) = T_0 + Ae^{-kt}.$$

Use the initial condition $T(0) = 200$ to determine A.

$$T(0) = T_0 + A = 200 \quad \rightarrow \quad A = 200 - T_0$$

Also, use the fact that the room temperature is $T_0 = 70$.

$$T(t) = 70 + 130e^{-kt}$$

After 1 minute, the coffee has cooled to 190°F.

$$T(1) = 70 + 130e^{-k} = 190$$

Solve for k.

$$130e^{-k} = 120$$
$$e^{-k} = \frac{12}{13}$$
$$\ln e^{-k} = \ln\frac{12}{13}$$
$$-k = \ln\frac{12}{13}$$
$$k = \ln\frac{13}{12}\ \frac{1}{\text{min}} \approx 0.0800\ \frac{1}{\text{min}}$$

So then

$$T(t) = 70 + 130\exp\left(-t\ln\frac{13}{12}\right).$$

Set $T(t) = 150$ and solve the resulting equation for t to find when the coffee reaches 150°F.

$$150 = 70 + 130\exp\left(-t\ln\frac{13}{12}\right)$$
$$80 = 130\exp\left(-t\ln\frac{13}{12}\right)$$
$$\exp\left(-t\ln\frac{13}{12}\right) = \frac{8}{13}$$
$$\ln\exp\left(-t\ln\frac{13}{12}\right) = \ln\frac{8}{13}$$
$$-t\ln\frac{13}{12} = \ln\frac{8}{13}$$
$$t = \frac{\ln\frac{13}{8}}{\ln\frac{13}{12}} \approx 6.07\text{ min}$$

Therefore, it takes about 6 minutes and 4 seconds for the coffee to cool to 150°F.