Problem 21

Assume that the conditions are as in Problem 20 except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

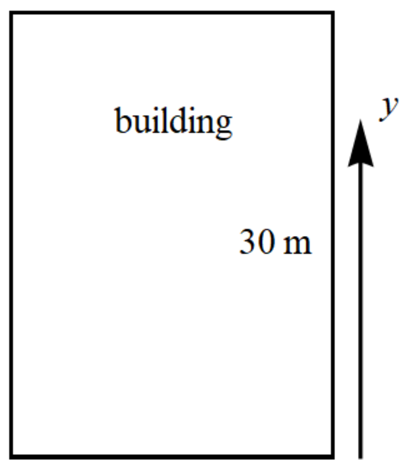
(a) Find the maximum height above the ground that the ball reaches.

(b) Find the time that the ball hits the ground.

(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 20.

Solution

Since we're taking air resistance into account now, the acceleration of the ball is no longer constant. That means the kinematic formulas cannot be applied as they were in Problem 20. The same coordinate system will be used, however.



Use Newton's second law to obtain the equation of motion.

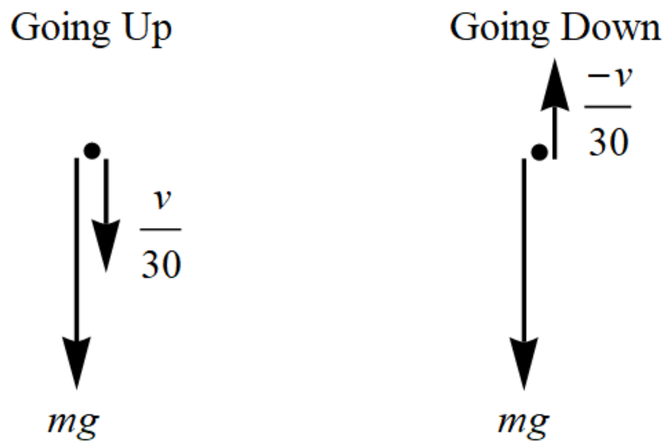
$$\sum \mathbf{F} = m\mathbf{a}$$

This is a vector equation; it consists of one equation for each direction in the coordinate system. Because only the y-direction is relevant, we just have

$$\sum F_y = ma_y.$$

Split up the ball's motion into two parts: the motion going up and the motion going down.

Below are the free-body diagrams for the two cases.



The equations of motion can now be written.

$$\text{Going Up:} \qquad -mg - \frac{v}{30} = ma_y$$

$$\text{Going Down:} \qquad -mg + \left(-\frac{v}{30}\right) = ma_y$$

The ODE for both parts is actually the same, so just one equation needs to be solved. Replace a_y with dv/dt.

$$m\frac{dv}{dt} = -mg - \frac{v}{30}$$

Bring $v/30$ to the left side and divide both sides by m.

$$\frac{dv}{dt} + \frac{v}{30m} = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{1}{30m}\, ds\right) = e^{t/(30m)}$$

Proceed with the multiplication.

$$e^{t/(30m)}\frac{dv}{dt} + \frac{v}{30m}e^{t/(30m)} = -ge^{t/(30m)}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{t/(30m)}v) = -ge^{t/(30m)}$$

Integrate both sides with respect to t.

$$e^{t/(30m)}v = -30mge^{t/(30m)} + C_1$$

Divide both sides by $e^{t/(30m)}$.

$$v(t) = -30mg + C_1e^{-t/(30m)}$$

Use the initial condition $v(0) = 20$ m/s to determine C_1.

$$v(0) = -30mg + C_1 = 20 \quad \rightarrow \quad C_1 = 20 + 30mg$$

So then the velocity (in meters per second) is

$$v(t) = -30mg + (20 + 30mg)e^{-t/(30m)}.$$

Integrate the velocity with respect to t to get the position.

$$\begin{aligned} y(t) &= \int v(t)\,dt \\ &= -30mgt - 30m(20 + 30mg)e^{-t/(30m)} + C_2 \end{aligned}$$

Use the initial condition $y(0) = 30$ meters to determine C_2.

$$y(0) = -30m(20 + 30mg) + C_2 = 30 \quad \rightarrow \quad C_2 = 30 + 30m(20 + 30mg)$$

The position (in meters) is then

$$y(t) = -30mgt - 30m(20 + 30mg)e^{-t/(30m)} + 30 + 30m(20 + 30mg).$$

Substitute $m = 0.15$ kilograms and $g = 9.81$ m/s^2 in both $y(t)$ and $v(t)$.

$$\begin{aligned} y(t) &\approx 318.7 - 44.15t - 288.7e^{-0.2222t} \\ v(t) &\approx -44.15 + 64.15e^{-0.2222t} \end{aligned}$$

Solve $v(t) = 0$ for t to find when the ball reaches its maximum height.

$$0 \approx -44.15 + 64.15e^{-0.2222t} \quad \rightarrow \quad t \approx 1.68 \text{ s}$$

Substitute this time into y to find the maximum height.

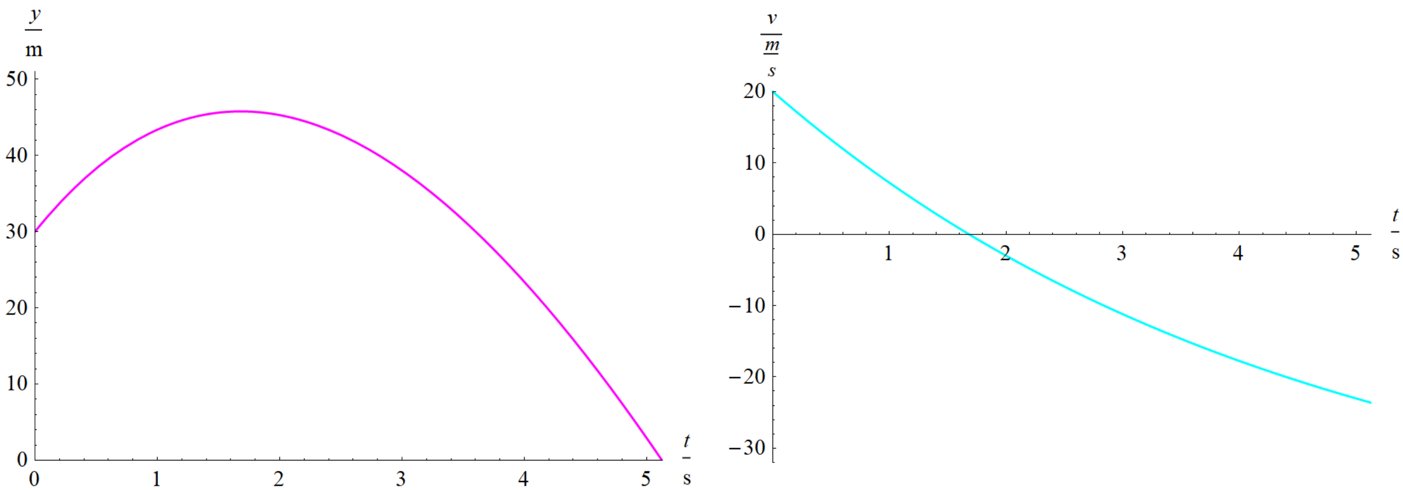
$$y_{\text{max}} \approx y(1.68) \approx 45.8 \text{ meters}$$

Then solve $y(t) = 0$ for t to determine when the ball hits the floor.

$$0 \approx 318.7 - 44.15t - 288.7e^{-0.2222t}$$

Graph the function on the right and find where it crosses the t-axis: $t \approx 5.12$ s.

Here the position and velocity of the ball subject to air resistance $|v|/30$ are plotted versus time.



In this next figure, the previous plots are superimposed with the plots of Problem 20, where there is no air resistance.