Problem 30

Let $v(t)$ and $w(t)$ be the horizontal and vertical components, respectively, of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$dv/dt = 0, \qquad dw/dt = -g.$$

(a) Show that

$$v = u\cos A, \qquad w = -gt + u\sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.

(b) Let $x(t)$ and $y(t)$ be the horizontal and vertical coordinates, respectively, of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t.

(c) Let $g = 32$ ft/s^2, $u = 125$ ft/s, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A; that is, plot $x(t)$ and $y(t)$ parametrically.

(d) Suppose the outfield wall is at a distance L and has height H. Find a relation between u and A that must be satisfied if the ball is to clear the wall.

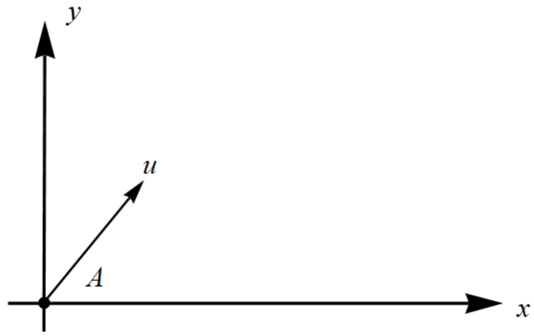
(e) Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part (d), find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/s.

(f) For $L = 350$ and $H = 10$, find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

[TYPO: These should read $L = 350$ ft and $H = 10$ ft.]

Solution

Below is a figure of the baseball with initial speed u at an angle A.



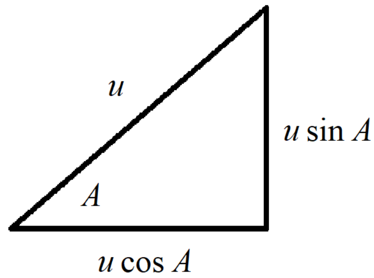
Part (a)

Integrate both ODEs with respect to t.

$$\frac{dv}{dt} = 0 \qquad\qquad \frac{dw}{dt} = -g$$

$$v(t) = C_1 \qquad\qquad w(t) = -gt + C_2$$

Decompose the initial velocity vector into its components along the x- and y-axes.



Use these components to determine C_1 and C_2.

$$v(0) = C_1 = u\cos A \qquad\qquad w(0) = C_2 = u\sin A$$

Therefore,

$$v(t) = u\cos A \qquad\qquad w(t) = -gt + u\sin A.$$

Part (b)

Integrate the velocities to get the positions.

$$\begin{aligned} x(t) &= \int v(t)\,dt = (u\cos A)t + C_3 \\ y(t) &= \int w(t)\,dt = -\frac{1}{2}gt^2 + (u\sin A)t + C_4 \end{aligned}$$

Use the initial conditions, $x(0) = 0$ and $y(0) = h$, to determine C_3 and C_4.

$$\begin{aligned} x(0) &= C_3 = 0 \\ y(0) &= C_4 = h \end{aligned}$$

Therefore,

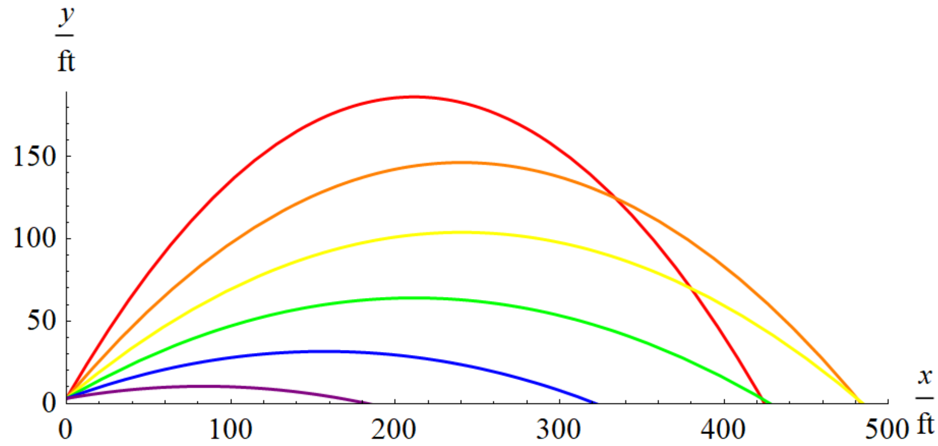
$$\begin{aligned} x(t) &= (u\cos A)t \\ y(t) &= -\frac{1}{2}gt^2 + (u\sin A)t + h. \end{aligned}$$

Part (c)

Let $g = 32$ ft/s^2, $u = 125$ ft/s, and $h = 3$ ft.

$$\begin{aligned} x(t) &= (125\cos A)t \\ y(t) &= -16t^2 + (125\sin A)t + 3. \end{aligned}$$

Below are graphs for $A = 10°$, $A = 20°$, $A = 30°$, $A = 40°$, $A = 50°$, and $A = 60°$ in purple, blue, green, yellow, orange, and red, respectively.



Part (d)

Set $x(t) = L$ and $y(t) \geq H$.

$$L = (u \cos A)t$$
$$H \leq -\frac{1}{2}gt^2 + (u \sin A)t + h$$

Since we just want a formula involving u and A, we will eliminate t. Solve the first equation for it.

$$t = \frac{L}{u \cos A}$$

Substitute this result into the inequality.

$$H \leq -\frac{1}{2}g\left(\frac{L}{u \cos A}\right)^2 + (u \sin A)\left(\frac{L}{u \cos A}\right) + h$$

Therefore,

$$H \leq -\frac{gL^2}{2u^2}\sec^2 A + L \tan A + h.$$

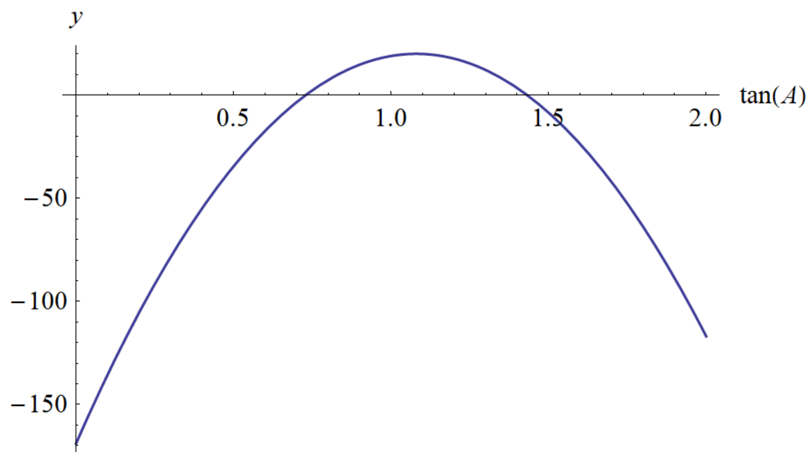
Part (e)

Set $H = 10$ ft, $L = 350$ ft, $g = 32$ ft/s^2, $u = 110$ ft/s, and $h = 3$ ft in the result of part (d).

$$10 \leq -\frac{(32)(350)^2}{2(110)^2}\sec^2 A + 350 \tan A + 3$$

$$-\frac{(32)(350)^2}{2(110)^2}(\tan^2 A + 1) + 350 \tan A - 7 \geq 0$$
$$-\frac{19\,600}{121}\tan^2 A + 350 \tan A - \frac{20\,447}{121} \geq 0$$

Below is a graph of the function on the left side versus $\tan A$. We care about the part of the parabola that lies above the horizontal axis.



Use the quadratic formula to locate the zeros.

$$\frac{-350+\sqrt{350^2-4\left(\frac{19\,600}{121}\right)\left(\frac{20\,447}{121}\right)}}{2\left(-\frac{19\,600}{121}\right)} \le \tan A \le \frac{-350-\sqrt{350^2-4\left(\frac{19\,600}{121}\right)\left(\frac{20\,447}{121}\right)}}{2\left(-\frac{19\,600}{121}\right)}$$

$$0.7283 \lesssim \tan A \lesssim 1.4324$$

$$\tan^{-1} 0.7283 \lesssim A \lesssim \tan^{-1} 1.4324$$

Therefore, in radians

$$0.6295 \lesssim A \lesssim 0.9613,$$

or in degrees

$$36.07° \lesssim A \lesssim 55.08°.$$

Part (f)

From the result of part (b), the position of the baseball is given by

$$x(t) = (u\cos A)t$$
$$y(t) = -\frac{1}{2}gt^2 + (u\sin A)t + h.$$

Set $x(t) = L$ and $y(t) = H$.

$$L = (u\cos A)t$$
$$H = -\frac{1}{2}gt^2 + (u\sin A)t + h.$$

Solve the first equation for t

$$t = \frac{L}{u\cos A}$$

and then substitute it into the second equation.

$$\begin{aligned} H &= -\frac{1}{2}g\left(\frac{L}{u\cos A}\right)^2 + (u\sin A)\left(\frac{L}{u\cos A}\right) + h \\ &= -\frac{gL^2}{2u^2\cos^2 A} + L\tan A + h \end{aligned}$$

Solve for u.

$$\frac{gL^2}{2u^2\cos^2 A} = L\tan A + h - H$$

$$\frac{gL^2}{2u^2} = L\tan A\cos^2 A + (h-H)\cos^2 A$$

$$\frac{1}{u^2} = \frac{2}{gL^2}[L\sin A\cos A + (h-H)\cos^2 A]$$

$$\begin{aligned} u &= \sqrt{\frac{gL^2}{2}\frac{1}{L\sin A\cos A + (h-H)\cos^2 A}} \\ &= \sqrt{\frac{gL^2}{2L\sin A\cos A + 2(h-H)\cos^2 A}} \\ &= \sqrt{\frac{gL^2}{L\sin 2A + 2(h-H)\cos^2 A}} \end{aligned}$$

Plug in $g = 32$ ft/s^2, $L = 350$ ft, $H = 10$ ft, and $h = 3$ ft.

$$u = 200\sqrt{7}(25\sin 2A - \cos^2 A)^{-1/2} \tag{1}$$

Differentiate u with respect to A and then set it equal to zero to find the values of A that extremize u.

$$\frac{du}{dA} = -100\sqrt{7}(25\sin 2A - \cos^2 A)^{-3/2}[50\cos 2A - 2\cos A(-\sin A)] = 0$$

$$-100\sqrt{7}(25\sin 2A - \cos^2 A)^{-3/2}(50\cos 2A + 2\sin A\cos A) = 0$$

$$50\cos 2A + 2\sin A\cos A = 0$$

$$50\cos 2A + \sin 2A = 0$$

$$50\cos 2A = -\sin 2A$$

$$2500\cos^2 2A = \sin^2 2A$$

$$2500\cos^2 2A = 1 - \cos^2 2A$$

$$2501\cos^2 2A = 1$$

$$\cos^2 2A = \frac{1}{2501}$$

$$\cos 2A = \pm\frac{1}{\sqrt{2501}}$$

$$2A = \left\{\pm\cos^{-1}\frac{1}{\sqrt{2501}}, \pm\cos^{-1}\frac{-1}{\sqrt{2501}}\right\}$$

The two minus signs can be discarded because they don't result in values of A between 0 and 90°.

$$2A = \left\{\cos^{-1}\frac{1}{\sqrt{2501}}, \cos^{-1}\frac{-1}{\sqrt{2501}}\right\}$$

As a result, the optimal angles that extremize u are

$$A = \left\{\frac{1}{2}\cos^{-1}\frac{1}{\sqrt{2501}}, \frac{1}{2}\cos^{-1}\frac{-1}{\sqrt{2501}}\right\} \approx \{0.775399, 0.795397\}.$$

Plug these values of A into equation (1) to find out the initial speeds associated with these angles.

$$u(A \approx 0.775399) \approx 106.937$$
$$u(A \approx 0.795397) \approx 106.894$$

Therefore, the minimum initial speed that a baseball can have to get to (L, H) is about 106.894 ft/s, and the angle from the horizontal it has to have is 0.795397 radians, or 45.57°.