Problem 32

Brachistochrone Problem. One of the famous problems in the history of mathematics is the brachistochrone[9] problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hôpital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown in Figure 2.3.6. The lower point Q has coordinates (x_0, y_0). It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2, \tag{i}$$

where k^2 is a certain positive constant to be determined later.

(a) Solve Eq. (i) for y'. Why is it necessary to choose the positive square root?

(b) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t. \tag{ii}$$

Show that the equation found in part (a) then takes the form

$$2k^2 \sin^2 t \, dt = dx. \tag{iii}$$

(c) Letting $\theta = 2t$, show that the solution of Eq. (iii) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin\theta)/2, \qquad y = k^2(1 - \cos\theta)/2. \tag{iv}$$

Equations (iv) are parametric equations of the solution of Eq. (i) that passes through $(0, 0)$. The graph of Eqs. (iv) is called a **cycloid**.

(d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$.

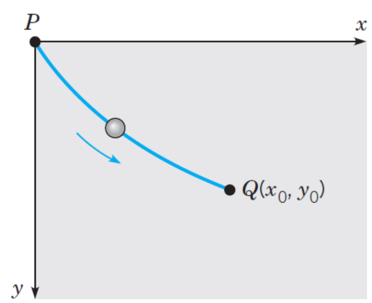


FIGURE 2.3.6 The brachistochrone.

Solution

[9]The word "brachistochrone" comes from the Greek words *brachistos*, meaning shortest, and *chronos*, meaning time.

Part (a)

$$(1 + y'^2)y = k^2$$

Divide both sides by y.

$$1 + y'^2 = \frac{k^2}{y}$$

Subtract 1 from both sides.

$$y'^2 = \frac{k^2}{y} - 1$$

Take the square root of both sides.

$$y' = \pm\sqrt{\frac{k^2}{y} - 1}$$

We choose the positive sign for y' (the slope) because Q is lower than P and the positive y-axis points downward. Therefore,

$$y' = \sqrt{\frac{k^2}{y} - 1}.$$

Part (b)

Let $y = k^2 \sin^2 t$. Differentiate both sides with respect to t.

$$\frac{dy}{dt} = 2k^2 \sin t \cos t$$

Use the chain rule for dy/dt, since y is a function of x.

$$\frac{dy}{dx}\frac{dx}{dt} = 2k^2 \sin t \cos t$$

Substitute the result of part (a) for dy/dx.

$$\sqrt{\frac{k^2}{y} - 1}\frac{dx}{dt} = 2k^2 \sin t \cos t$$

Substitute $y = k^2 \sin^2 t$ and simplify both sides.

$$\sqrt{\frac{k^2}{k^2 \sin^2 t} - 1}\frac{dx}{dt} = 2k^2 \sin t \cos t$$

$$\sqrt{\csc^2 t - 1}\frac{dx}{dt} = 2k^2 \sin t \cos t$$

$$\sqrt{\cot^2 t}\frac{dx}{dt} = 2k^2 \sin t \cos t$$

$$(\cot t)\frac{dx}{dt} = 2k^2 \sin t \cos t$$

$$\frac{dx}{dt} = 2k^2 \sin^2 t$$

Therefore,

$$dx = 2k^2 \sin^2 t \, dt.$$

Part (c)

Continue from the result of part (b).

$$dx = 2k^2 \sin^2 t \, dt$$

Integrate both sides.

$$\begin{aligned} x &= \int 2k^2 \sin^2 t \, dt \\ &= 2k^2 \int \sin^2 t \, dt \\ &= 2k^2 \int \frac{1}{2}(1 - \cos 2t) \, dt \\ &= k^2 \left(t - \frac{1}{2} \sin 2t \right) + C_1 \end{aligned}$$

We also have

$$\begin{aligned} y &= k^2 \sin^2 t \\ &= \frac{k^2}{2}(1 - \cos 2t). \end{aligned}$$

In order to have $x = 0$ when $y = 0$ (that is, $t = 0$), C_1 must be zero: $C_1 = 0$.

$$\begin{aligned} x(t) &= k^2 \left(t - \frac{1}{2} \sin 2t \right) \\ y(t) &= \frac{k^2}{2}(1 - \cos 2t) \end{aligned}$$

Now replace $\theta = 2t$ to get the desired result.

$$\begin{aligned} x(t) &= k^2 \left(\frac{\theta}{2} - \frac{1}{2} \sin \theta \right) \\ y(t) &= \frac{k^2}{2}(1 - \cos \theta) \end{aligned}$$

Therefore,

$$\begin{aligned} x(t) &= \frac{k^2}{2}(\theta - \sin \theta) \\ y(t) &= \frac{k^2}{2}(1 - \cos \theta). \end{aligned}$$

Part (d)

The cycloid has to go through the point $(x = 1, y = 2)$ at some value of θ, say θ_0. Solve the resulting system of equations for k and θ_0.

$$1 = \frac{k^2}{2}(\theta_0 - \sin\theta_0) \tag{1}$$

$$2 = \frac{k^2}{2}(1 - \cos\theta_0). \tag{2}$$

Multiply both sides of equation (1) by 2.

$$2 = k^2(\theta_0 - \sin\theta_0)$$

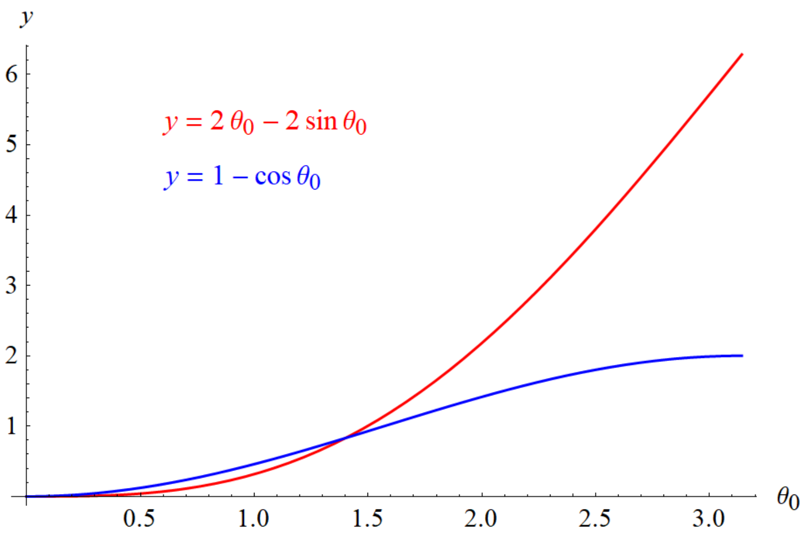
Then substitute the right side of equation (2) for 2.

$$\frac{k^2}{2}(1 - \cos\theta_0) = k^2(\theta_0 - \sin\theta_0)$$

Multiply both sides by $2/k^2$.

$$1 - \cos\theta_0 = 2\theta_0 - 2\sin\theta_0$$

Graph the function on each side and find what value of θ_0 they intersect.



We see that $\theta_0 \approx 1.4$. Plug this into either equation (1) or equation (2) to determine k.

$$2 \approx \frac{k^2}{2}(1 - \cos 1.4)$$

Therefore,

$$k \approx \sqrt{\frac{4}{1 - \cos 1.4}} \approx 2.193.$$