Problem 1

Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

Solution

Let t represent the time in minutes, let $V = V(t)$ represent the volume in liters, and let $m = m(t)$ represent the mass of dye in grams. The tank initially contains 200 L and has a concentration, defined as mass of dye per unit volume, of 1 g/L.

$$V(0) = 200\text{ L}$$
$$\frac{m(0)}{V(0)} = 1\ \frac{\text{g}}{\text{L}} \quad \rightarrow \quad \frac{m(0)}{200\text{ L}} = 1\ \frac{\text{g}}{\text{L}} \quad \rightarrow \quad m(0) = 200\text{ g}$$

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time.

$$\begin{aligned}\frac{dV}{dt} &= 2\ \frac{\text{L}}{\text{min}} - 2\ \frac{\text{L}}{\text{min}} \\ &= 0\end{aligned}$$

Integrate both sides with respect to t.

$$V(t) = C_1$$

Use the initial condition for V to determine C_1.

$$V(0) = C_1 = 200\text{ L}$$

So the volume is

$$V(t) = 200\text{ L}.$$

Now apply the law to the mass, noting that dm/dt is the rate that the mass of dye increases with respect to time. To obtain the rate of mass flow, multiply the concentration by the volume flow rate. Since fresh water is flowing in, the concentration of dye flowing in is 0 g/L. In addition, the concentration flowing out is $m(t)/V(t)$ because the solution is well-stirred.

$$\begin{aligned}\frac{dm}{dt} &= \left(2\ \frac{\text{L}}{\text{min}}\right)\left(0\ \frac{\text{g}}{\text{L}}\right) - \left(2\ \frac{\text{L}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= -\frac{m}{100}\end{aligned}$$

Separate variables to solve this ODE.

$$\frac{dm}{m} = -\frac{dt}{100}$$

Integrate both sides.

$$\ln|m| = -\frac{t}{100} + C_2$$

Exponentiate both sides.

$$\begin{aligned} |m| &= e^{-t/100+C_2} \\ &= e^{C_2}e^{-t/100} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$m(t) = \pm e^{C_2}e^{-t/100}$$

Use a new constant of integration A for $\pm e^{C_2}$.

$$m(t) = Ae^{-t/100}$$

Use the initial condition for m to determine A.

$$m(0) = A = 200 \text{ g}$$

So the mass of dye in the tank is

$$m(t) = 200e^{-t/100} \text{ g}.$$

Our aim now is to find the time at which the concentration of dye reaches 1% of its original value. Solve the following equation for t.

$$\begin{aligned} \frac{m(t)}{V(t)} &= 0.01\left(1 \, \frac{\text{g}}{\text{L}}\right) \\ \frac{200e^{-t/100} \text{ g}}{200 \text{ L}} &= 0.01 \, \frac{\text{g}}{\text{L}} \\ e^{-t/100} &= 0.01 \\ \ln e^{-t/100} &= \ln 0.01 \\ -\frac{t}{100} &= -\ln 100 \\ t &= 100 \ln 100 \text{ min} \approx 460.52 \text{ min} \end{aligned}$$

Therefore, it takes approximately 7 hours, 40 minutes, and 31 seconds for the concentration to reach 1% of its original value.