Problem 18

Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, u satisfies the differential equation

$$\frac{du}{dt} = -k[u - T(t)], \tag{i}$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos \omega t$.

(a) Solve Eq. (i) and express $u(t)$ in terms of t, k, T_0, T_1, and ω. Observe that part of your solution approaches zero as t becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

(b) Suppose that t is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60°\text{F}$, $T_1 = 15°\text{F}$, and $k = 0.2/\text{h}$. Draw graphs of $S(t)$ and $T(t)$ versus t on the same axes. From your graph estimate the amplitude R of the oscillatory part of $S(t)$. Also estimate the time lag τ between corresponding maxima of $T(t)$ and $S(t)$.

(c) Let k, T_0, T_1, and ω now be unspecified. Write the oscillatory part of $S(t)$ in the form $R\cos[\omega(t - \tau)]$. Use trigonometric identities to find expressions for R and τ. Let T_1 and ω have the values given in part (b), and plot graphs of R and τ versus k.

Solution

Part (a)

Distribute k in the ODE.

$$\frac{du}{dt} = -ku + kT(t)$$

Bring ku to the left side and substitute $T(t) = T_0 + T_1 \cos \omega t$.

$$\frac{du}{dt} + ku = kT_0 + kT_1 \cos \omega t$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t k\, ds\right) = e^{kt}$$

Proceed with the multiplication.

$$e^{kt}\frac{du}{dt} + ke^{kt}u = kT_0e^{kt} + kT_1e^{kt}\cos \omega t$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}(e^{kt}u) = kT_0e^{kt} + kT_1e^{kt}\cos \omega t$$

Integrate both sides with respect to t.

$$e^{kt}u = \int^t (kT_0e^{ks} + kT_1e^{ks}\cos \omega s)\, ds + C$$

Distribute the integral and bring the constants in front of them.

$$e^{kt}u = T_0 e^{kt} + kT_1 \int^t e^{ks} \cos \omega s \, ds + C \tag{1}$$

Use integration by parts twice for the remaining integral.

$$\begin{aligned}
\int^t e^{ks} \cos \omega s \, ds &= \int^t e^{ks} \frac{d}{ds}\left(\frac{1}{\omega}\sin \omega s\right) ds \\
&= e^{ks}\left(\frac{1}{\omega}\sin \omega s\right)\Big|^t - \int^t k e^{ks} \frac{1}{\omega} \sin \omega s \, ds \\
&= \frac{e^{kt}\sin \omega t}{\omega} - \frac{k}{\omega}\int^t e^{ks}\frac{d}{ds}\left(-\frac{1}{\omega}\cos \omega s\right) ds \\
&= \frac{e^{kt}\sin \omega t}{\omega} - \frac{k}{\omega}\left[e^{ks}\left(-\frac{1}{\omega}\cos \omega s\right)\Big|^t - \int^t k e^{ks}\left(-\frac{1}{\omega}\cos \omega s\right) ds\right] \\
&= \frac{e^{kt}\sin \omega t}{\omega} - \frac{k}{\omega}\left[e^{kt}\left(-\frac{1}{\omega}\cos \omega t\right) + \frac{k}{\omega}\int^t e^{ks}\cos \omega s \, ds\right] \\
&= \frac{e^{kt}\sin \omega t}{\omega} + \frac{k}{\omega^2}e^{kt}\cos \omega t - \frac{k^2}{\omega^2}\int^t e^{ks}\cos \omega s \, ds
\end{aligned}$$

Solve for the integral.

$$\left(1 + \frac{k^2}{\omega^2}\right)\int^t e^{ks}\cos \omega s \, ds = \frac{e^{kt}}{\omega^2}(k \cos \omega t + \omega \sin \omega t)$$

As a result,

$$\int^t e^{ks}\cos \omega s \, ds = \frac{e^{kt}}{k^2 + \omega^2}(k \cos \omega t + \omega \sin \omega t).$$

Substitute this result into equation (1).

$$e^{kt}u = T_0 e^{kt} + kT_1 \frac{e^{kt}}{k^2+\omega^2}(k \cos \omega t + \omega \sin \omega t) + C$$

Divide both sides by e^{kt} to get the general solution.

$$u(t) = \underbrace{T_0 + T_1 \frac{k}{k^2+\omega^2}(k\cos \omega t + \omega \sin \omega t)}_{\text{steady}} + \underbrace{Ce^{-kt}}_{\text{transient}}$$

It can be thought of as the sum of a steady term, one that remains as $t \to \infty$, and a transient term, one that dies out as $t \to \infty$. Define $S(t)$ to be the steady part.

$$S(t) = T_0 + T_1 \frac{k}{k^2+\omega^2}(k \cos \omega t + \omega \sin \omega t)$$

Part (b)

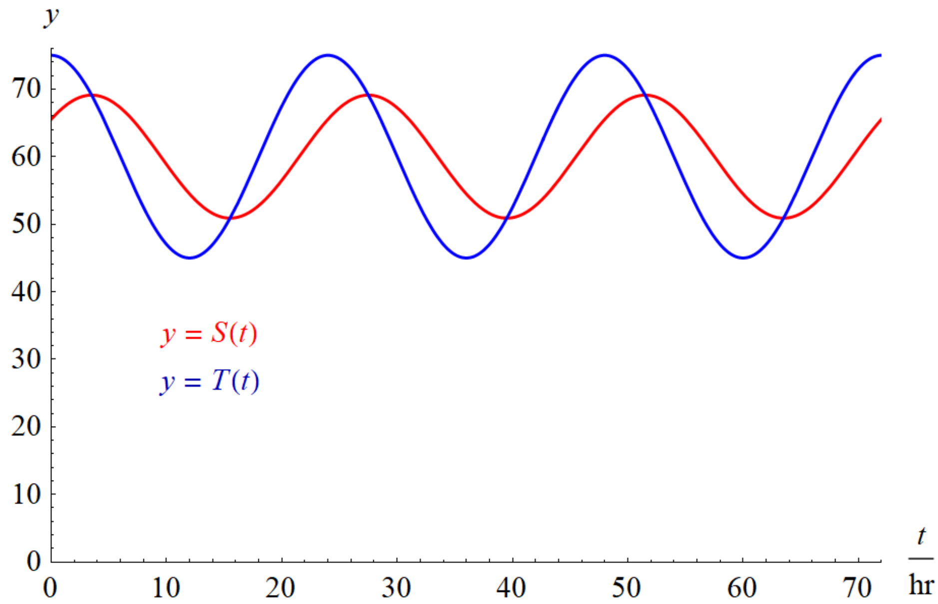
Let $\omega = \pi/12$, $T_0 = 60$, $T_1 = 15$, and $k = 0.2$. Then

$$\begin{aligned} S(t) &= 60 + 15\frac{0.2}{0.04 + \frac{\pi^2}{144}}\left(0.2\cos\frac{\pi t}{12} + \frac{\pi}{12}\sin\frac{\pi t}{12}\right) \\ &= 60 + \frac{3}{5\left(\frac{1}{25} + \frac{\pi^2}{144}\right)}\cos\frac{\pi t}{12} + \frac{\pi}{4\left(\frac{1}{25} + \frac{\pi^2}{144}\right)}\sin\frac{\pi t}{12} \end{aligned}$$

and

$$T(t) = 60 + 15\cos\frac{\pi t}{12}.$$

Below is a plot of $S(t)$ and $T(t)$ versus t.



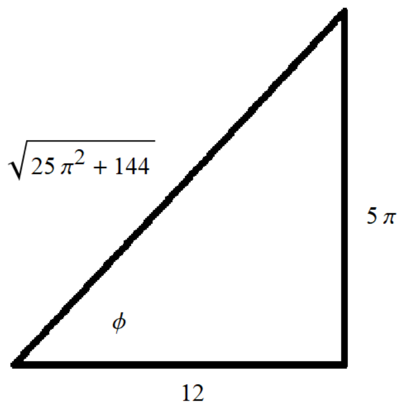
There's no need to estimate the amplitude from the graph. Let

$$\begin{aligned} A\cos\phi &= \frac{3}{5\left(\frac{1}{25} + \frac{\pi^2}{144}\right)} \\ A\sin\phi &= \frac{\pi}{4\left(\frac{1}{25} + \frac{\pi^2}{144}\right)} \end{aligned}$$

and solve the two equations for the two unknowns, A and ϕ. Divide both sides of the second equation by those of the first.

$$\tan\phi = \frac{5\pi}{12}$$

Draw the implied triangle to determine $\cos\phi$.



Now A can be found.

$$A\left(\frac{12}{\sqrt{25\pi^2+144}}\right)=\frac{3}{5\left(\frac{1}{25}+\frac{\pi^2}{144}\right)} \quad \rightarrow \quad A=\frac{180}{\sqrt{25\pi^2+144}}$$

Consequently,

$$\begin{aligned} S(t) &= 60 + A\cos\phi\cos\frac{\pi t}{12} + A\sin\phi\sin\frac{\pi t}{12} \\ &= 60 + A\cos\left(\frac{\pi t}{12}-\phi\right) \\ &= 60 + \frac{180}{\sqrt{25\pi^2+144}}\cos\left(\frac{\pi t}{12}-\tan^{-1}\frac{5\pi}{12}\right) \\ &= 60 + \frac{180}{\sqrt{25\pi^2+144}}\cos\left[\frac{\pi}{12}\left(t-\frac{12}{\pi}\tan^{-1}\frac{5\pi}{12}\right)\right]. \end{aligned}$$

The amplitude is

$$R=\frac{180}{\sqrt{25\pi^2+144}}\approx 9.11°\text{F},$$

and the graph of $T(t)$ lags behind that of $S(t)$ by

$$\tau=\frac{12}{\pi}\tan^{-1}\frac{5\pi}{12}\approx 3.51 \text{ hours.}$$

Part (c)

Recall that $S(t)$ is the steady part of the temperature.

$$\begin{aligned} S(t) &= T_0 + T_1 \frac{k}{k^2+\omega^2}(k\cos\omega t + \omega \sin\omega t) \\ &= T_0 + T_1 \frac{k^2}{k^2+\omega^2}\cos\omega t + T_1 \frac{k\omega}{k^2+\omega^2}\sin\omega t \end{aligned}$$

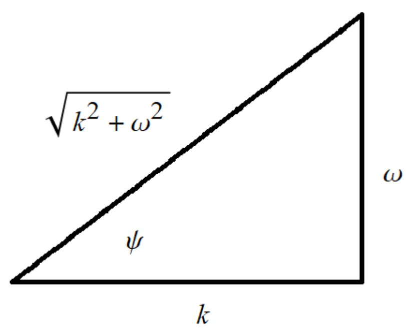
Let

$$\begin{aligned} R\cos\psi &= T_1 \frac{k^2}{k^2+\omega^2} \\ R\sin\psi &= T_1 \frac{k\omega}{k^2+\omega^2} \end{aligned}$$

and solve the two equations for the two unknowns, R and ψ. Divide both sides of the second equation by those of the first.

$$\tan\psi = \frac{\omega}{k}$$

Draw the implied triangle to determine $\cos\psi$.



Now R can be found.

$$R\left(\frac{k}{\sqrt{k^2+\omega^2}}\right) = T_1 \frac{k^2}{k^2+\omega^2} \quad \rightarrow \quad R = T_1 \frac{k}{\sqrt{k^2+\omega^2}}$$

Therefore,

$$\begin{aligned} S(t) &= T_0 + R\cos\psi\cos\omega t + R\sin\psi\sin\omega t \\ &= T_0 + R\cos(\omega t - \psi) \\ &= T_0 + T_1 \frac{k}{\sqrt{k^2+\omega^2}}\cos\left(\omega t - \tan^{-1}\frac{\omega}{k}\right) \\ &= T_0 + T_1 \frac{k}{\sqrt{k^2+\omega^2}}\cos\left[\omega\left(t - \frac{1}{\omega}\tan^{-1}\frac{\omega}{k}\right)\right] \end{aligned}$$

and

$$\tau = \frac{1}{\omega}\tan^{-1}\frac{\omega}{k}.$$

If $T_1 = 15$ and $\omega = \pi/12$, then

$$R = 15\frac{k}{\sqrt{k^2 + \frac{\pi^2}{144}}}$$
$$\tau = \frac{12}{\pi}\tan^{-1}\frac{\pi}{12k}.$$