Problem 18

Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, u satisfies the differential equation

$$\frac{du}{dt} = -k[u - T(t)], \tag{i}$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos \omega t$.

(a) Solve Eq. (i) and express $u(t)$ in terms of t, k, T_0, T_1, and ω. Observe that part of your solution approaches zero as t becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

(b) Suppose that t is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60$°F, $T_1 = 15$°F, and $k = 0.2$/h. Draw graphs of $S(t)$ and $T(t)$ versus t on the same axes. From your graph estimate the amplitude R of the oscillatory part of $S(t)$. Also estimate the time lag τ between corresponding maxima of $T(t)$ and $S(t)$.

(c) Let k, T_0, T_1, and ω now be unspecified. Write the oscillatory part of $S(t)$ in the form $R\cos[\omega(t - \tau)]$. Use trigonometric identities to find expressions for R and τ. Let T_1 and ω have the values given in part (b), and plot graphs of R and τ versus k.