Problem 19

Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P. Note that the given assumptions neglect a number of factors that may, in some cases, be important—for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are deposited unevenly throughout the lake but (usually) at isolated points around its periphery. The results below must be interpreted in the light of the neglect of such factors as these.

(a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \to \infty$?

(b) If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval T that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

(c) Table 2.3.2 contains data[6] for several of the Great Lakes. Using these data, determine from part (b) the time T that is needed to reduce the contamination of each of these lakes to 10% of the original value.

TABLE 2.3.2 Volume and Flow Data for the Great Lakes

Lake	V (km^3 $\times$ 10^3)	r (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

Solution

Part (a)

The conservation law that governs the mass of pollutant in the lake is as follows.

$$\text{rate of mass accumulation} = \text{rate of mass in} - \text{rate of mass out}$$

The rate of mass in is the sum of rk and P, and the rate of mass out is $rc(t)$.

$$\begin{aligned}\frac{dQ}{dt} &= rk + P - rc(t)\\ &= rk + P - \frac{r}{V}Q(t)\end{aligned}$$

[6]This problem is based on R. H. Rainey, "Natural Displacement of Pollution from the Great Lakes," *Science 155* (1967), pp. 1242–1243; the information in the table was taken from that source.

Bring the term with Q to the left side.

$$\frac{dQ}{dt} + \frac{r}{V}Q = rk + P$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{r}{V}\,ds\right) = e^{rt/V}$$

Proceed with the multiplication.

$$e^{rt/V}\frac{dQ}{dt} + \frac{r}{V}e^{rt/V}Q = rke^{rt/V} + Pe^{rt/V}$$

The left side can be written as $d/dt(IQ)$ by the product rule.

$$\frac{d}{dt}(e^{rt/V}Q) = rke^{rt/V} + Pe^{rt/V}$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{rt/V}Q &= \int^t (rke^{rs/V} + Pe^{rs/V})\,ds + C \\ &= rk\left(\frac{V}{r}\right)e^{rt/V} + P\left(\frac{V}{r}\right)e^{rt/V} + C \\ &= kVe^{rt/V} + \frac{PV}{r}e^{rt/V} + C \end{aligned}$$

Divide both sides by $e^{rt/V}$.

$$Q(t) = kV + \frac{PV}{r} + Ce^{-rt/V}$$

Divide both sides by V to obtain the concentration.

$$c(t) = \frac{Q(t)}{V} = k + \frac{P}{r} + \frac{C}{V}e^{-rt/V}$$

Apply the initial condition $c(0) = c_0$ to determine C.

$$c(0) = k + \frac{P}{r} + \frac{C}{V} = c_0 \quad \rightarrow \quad \frac{C}{V} = c_0 - k - \frac{P}{r}$$

Therefore, the concentration is

$$c(t) = \frac{Q(t)}{V} = k + \frac{P}{r} + \left(c_0 - k - \frac{P}{r}\right)e^{-rt/V}.$$

Because of the decaying exponential function, the limit of $c(t)$ as $t \to \infty$ is

$$\lim_{t\to\infty} c(t) = k + \frac{P}{r}.$$

Part (b)

If the pollution stops, then the rate of mass in becomes zero in the conservation law.

$$\text{rate of mass accumulation} = \underbrace{\text{rate of mass in}}_{=\,0} - \text{rate of mass out}$$

The rate of mass out is still $rc(t)$.

$$\begin{aligned}\frac{dQ}{dt} &= -rc(t)\\ &= -\frac{r}{V}Q(t)\end{aligned}$$

Divide both sides by Q.

$$\frac{\frac{dQ}{dt}}{Q} = -\frac{r}{V}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln Q = -\frac{r}{V}$$

Integrate both sides with respect to t.

$$\ln Q = -\frac{r}{V}t + C_1$$

Exponentiate both sides.

$$\begin{aligned}Q(t) &= e^{-rt/V + C_1}\\ &= e^{C_1}e^{-rt/V}\end{aligned}$$

Use a new constant A_1 for e^{C_1}.

$$Q(t) = A_1 e^{-rt/V}$$

Since Q is the mass of pollutant, divide both sides by V to get the concentration.

$$c(t) = \frac{Q(t)}{V} = \frac{A_1}{V}e^{-rt/V}$$

Use the initial condition $c(0) = c_0$ to determine A_1.

$$c(0) = \frac{A_1}{V} = c_0$$

Therefore,

$$c(t) = c_0 e^{-rt/V}.$$

Set $c(t) = 0.5c_0$ and solve for $t = T$ to find how long it will take for the concentration to reach half its initial value.

$$\begin{aligned}0.5c_0 &= c_0 e^{-rT/V}\\ e^{-rT/V} &= 0.5\\ \ln e^{-rT/V} &= \ln 0.5\end{aligned}$$

$$-\frac{rT}{V} = \ln\frac{1}{2}$$
$$T = \frac{V}{r}\ln 2$$

On the other hand, set $c(t) = 0.1c_0$ and solve for $t = T$ to find how long it will take for the concentration to reach one-tenth its initial value.

$$0.1c_0 = c_0 e^{-rT/V}$$
$$e^{-rT/V} = 0.1$$
$$\ln e^{-rT/V} = \ln 0.1$$
$$-\frac{rT}{V} = \ln\frac{1}{10}$$
$$T = \frac{V}{r}\ln 10$$

Part (c)

Plug in the numbers for V and r into the previous formula.

Lake Superior: $$T = \frac{V}{r}\ln 10 = \frac{12.2\times 10^3\ \text{km}^3}{65.2\ \frac{\text{km}^3}{\text{year}}}\ln 10 \approx 431 \text{ years}$$

Lake Michigan: $$T = \frac{V}{r}\ln 10 = \frac{4.9\times 10^3\ \text{km}^3}{158\ \frac{\text{km}^3}{\text{year}}}\ln 10 \approx 71.4 \text{ years}$$

Lake Erie: $$T = \frac{V}{r}\ln 10 = \frac{0.46\times 10^3\ \text{km}^3}{175\ \frac{\text{km}^3}{\text{year}}}\ln 10 \approx 6.05 \text{ years}$$

Lake Ontario: $$T = \frac{V}{r}\ln 10 = \frac{1.6\times 10^3\ \text{km}^3}{209\ \frac{\text{km}^3}{\text{year}}}\ln 10 \approx 17.6 \text{ years}$$