

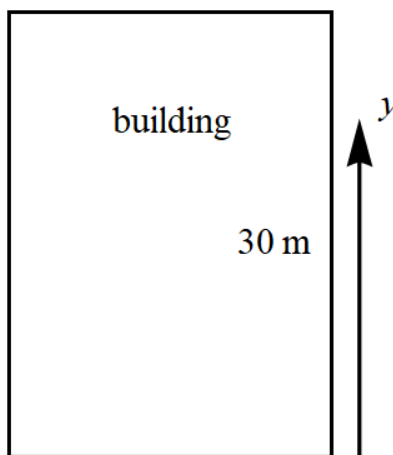
Problem 22

Assume that the conditions are as in Problem 20 except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.

- Find the maximum height above the ground that the ball reaches.
- Find the time that the ball hits the ground.
- Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 20 and 21.

Solution

Since we're taking air resistance into account now, the acceleration of the ball is no longer constant. That means the kinematic formulas cannot be applied as they were in Problem 20. The same coordinate system will be used, however.



Use Newton's second law to obtain the equation of motion.

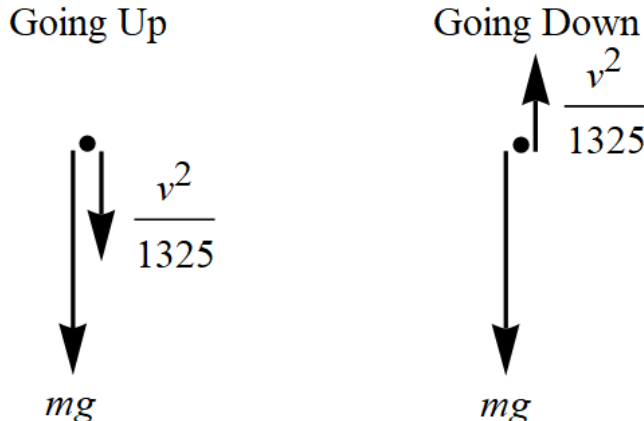
$$\sum \mathbf{F} = m\mathbf{a}$$

This is a vector equation; it consists of one equation for each direction in the coordinate system. Because only the y -direction is relevant, we just have

$$\sum F_y = ma_y.$$

Split up the ball's motion into two parts: the motion going up and the motion going down.

Below are the free-body diagrams for the two cases.



The equations of motion can now be written.

$$\begin{aligned} \text{Going Up:} \quad & -mg - \frac{v^2}{1325} = ma_y \\ \text{Going Down:} \quad & -mg + \frac{v^2}{1325} = ma_y \end{aligned}$$

Replace a_y with dv/dt in both equations and solve them both by separating variables.

Going Up

$$\begin{aligned} m \frac{dv}{dt} &= -\frac{v^2}{1325} - mg \\ m \frac{dv}{dt} &= -\frac{v^2 + 1325mg}{1325} \\ \frac{dv}{v^2 + 1325mg} &= -\frac{dt}{1325m} \\ \int \frac{dv}{v^2 + 1325mg} &= -\frac{t}{1325m} + C_1 \\ \frac{1}{\sqrt{1325mg}} \tan^{-1} \frac{v}{\sqrt{1325mg}} &= -\frac{t}{1325m} + C_1 \\ \tan^{-1} \frac{v}{\sqrt{1325mg}} &= -\sqrt{\frac{g}{1325m}} t + C_3 \\ \frac{v}{\sqrt{1325mg}} &= \tan \left(-\sqrt{\frac{g}{1325m}} t + C_3 \right) \\ v(t) &= \sqrt{1325mg} \tan \left(-\sqrt{\frac{g}{1325m}} t + C_3 \right) \end{aligned}$$

Going Down

$$\begin{aligned} m \frac{dv}{dt} &= -mg + \frac{v^2}{1325} \\ m \frac{dv}{dt} &= -\frac{1325mg - v^2}{1325} \\ \frac{dv}{1325mg - v^2} &= -\frac{dt}{1325m} \\ \int \frac{dv}{1325mg - v^2} &= -\frac{t}{1325m} + C_2 \\ \frac{1}{\sqrt{1325mg}} \tanh^{-1} \frac{v}{\sqrt{1325mg}} &= -\frac{t}{1325m} + C_2 \\ \tanh^{-1} \frac{v}{\sqrt{1325mg}} &= -\sqrt{\frac{g}{1325m}} t + C_4 \\ \frac{v}{\sqrt{1325mg}} &= \tanh \left(-\sqrt{\frac{g}{1325m}} t + C_4 \right) \\ v(t) &= \sqrt{1325mg} \tanh \left(-\sqrt{\frac{g}{1325m}} t + C_4 \right) \end{aligned}$$

Integrate the velocities to get the positions.

$$\begin{aligned} y(t) &= \frac{\sqrt{1325mg}}{\sqrt{\frac{g}{1325m}}} \ln \left| \cos \left(-\sqrt{\frac{g}{1325m}} t + C_3 \right) \right| + C_5 \\ y(t) &= -\frac{\sqrt{1325mg}}{\sqrt{\frac{g}{1325m}}} \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + C_4 \right) + C_6 \end{aligned}$$

Our task now is to determine the constants of integration, C_3 , C_4 , C_5 , and C_6 . Initially the ball is thrown up at a speed of 20 m/s. Apply $v(0) = 20$ to determine C_3 .

$$v(0) = \sqrt{1325mg} \tan(C_3) = 20 \quad \rightarrow \quad C_3 = \tan^{-1} \frac{20}{\sqrt{1325mg}}$$

So then

$$\text{Going Up: } v(t) = \sqrt{1325mg} \tan \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right).$$

Now we will find the time $t = t_{\max}$ at which the ball reaches its maximum height. This occurs when the velocity is zero, that is, when the argument of tangent is zero.

$$\text{Going Up: } v(t) = 0 \quad \Rightarrow \quad t_{\max} = \sqrt{\frac{1325m}{g}} \tan^{-1} \frac{20}{\sqrt{1325mg}}$$

At the maximum height, the velocity of the ball changes direction. Consequently, at t_{\max} , both velocities must be equal to zero. Use $v(t_{\max}) = 0$ to determine C_4 .

$$\begin{aligned} \text{Going Down: } v(t_{\max}) &= \sqrt{1325mg} \tanh \left(-\sqrt{\frac{g}{1325m}} t_{\max} + C_4 \right) = 0 \\ &= \sqrt{1325mg} \tanh \left(-\tan^{-1} \frac{20}{\sqrt{1325mg}} + C_4 \right) = 0 \quad \rightarrow \quad C_4 = \tan^{-1} \frac{20}{\sqrt{1325mg}} \end{aligned}$$

So then

$$\text{Going Down: } v(t) = \sqrt{1325mg} \tanh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right).$$

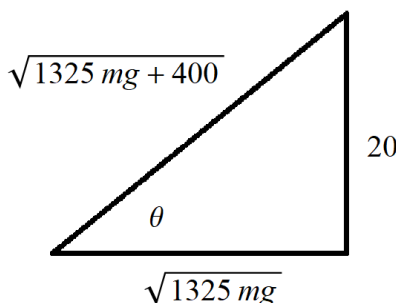
With C_3 and C_4 determined, the positions are then

$$\begin{aligned} \text{Going Up: } y(t) &= 1325m \ln \left| \cos \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) \right| + C_5 \\ \text{Going Down: } y(t) &= -1325m \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) + C_6. \end{aligned}$$

Initially the ball is 30 meters high, so apply $y(0) = 30$ to obtain C_5 .

$$y(0) = 1325m \ln \left| \cos \tan^{-1} \frac{20}{\sqrt{1325mg}} \right| + C_5 = 30$$

Draw the implied right triangle to find the cosine of $\theta = \tan^{-1}(20/\sqrt{1325mg})$.



We see that

$$\cos \tan^{-1} \frac{20}{\sqrt{1325mg}} = \frac{\sqrt{1325mg}}{\sqrt{1325mg + 400}} = \sqrt{\frac{53mg}{53mg + 16}},$$

which means

$$y(0) = 1325m \ln \sqrt{\frac{53mg}{53mg + 16}} + C_5 = 30 \quad \rightarrow \quad C_5 = 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}}.$$

So then

$$\text{Going Up: } y(t) = 1325m \ln \left| \cos \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) \right| + 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}}.$$

If we plug in t_{\max} to this result, we'll get the maximum height of the ball.

$$\text{Going Up: } y_{\max} = y(t_{\max}) = 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}}$$

Since the position must be continuous, use $y(t_{\max}) = y_{\max}$ to determine C_6 .

$$\text{Going Down: } y(t_{\max}) = C_6 = 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}}$$

So then

$$\text{Going Down: } y(t) = -1325m \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) + 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}}.$$

Set $y(t) = 0$ in this equation and solve for t to find how long it takes for the ball to hit the floor. (It's easier to just graph the function and see where the curve crosses the t -axis.)

$$\begin{aligned} 0 &= -1325m \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) + 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}} \\ 1325m \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) &= 30 + 1325m \ln \sqrt{1 + \frac{16}{53mg}} \\ \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) &= \frac{6}{265m} + \ln \sqrt{1 + \frac{16}{53mg}} \\ \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) &= \exp \left(\frac{6}{265m} \right) \sqrt{1 + \frac{16}{53mg}} \\ -\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} &= \pm \cosh^{-1} \left[\exp \left(\frac{6}{265m} \right) \sqrt{1 + \frac{16}{53mg}} \right] \\ t &= \sqrt{\frac{1325m}{g}} \left\{ \tan^{-1} \frac{20}{\sqrt{1325mg}} \mp \cosh^{-1} \left[\exp \left(\frac{6}{265m} \right) \sqrt{1 + \frac{16}{53mg}} \right] \right\} \end{aligned}$$

We choose the plus sign because t should be a positive number.

$$t = \sqrt{\frac{1325m}{g}} \left\{ \tan^{-1} \frac{20}{\sqrt{1325mg}} + \cosh^{-1} \left[\exp \left(\frac{6}{265m} \right) \sqrt{1 + \frac{16}{53mg}} \right] \right\}$$

Now plug in $m = 0.15$ kg and $g = 9.81$ m/s² to get numerical values for this and y_{\max} .

$$y_{\max} = 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}} \approx 48.5 \text{ meters}$$

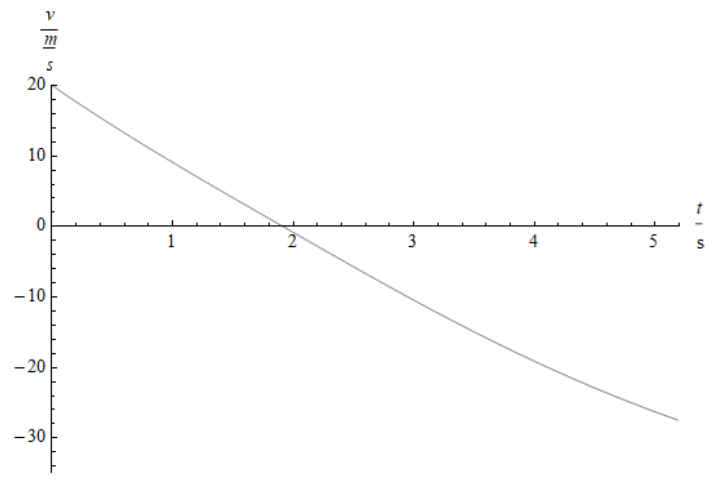
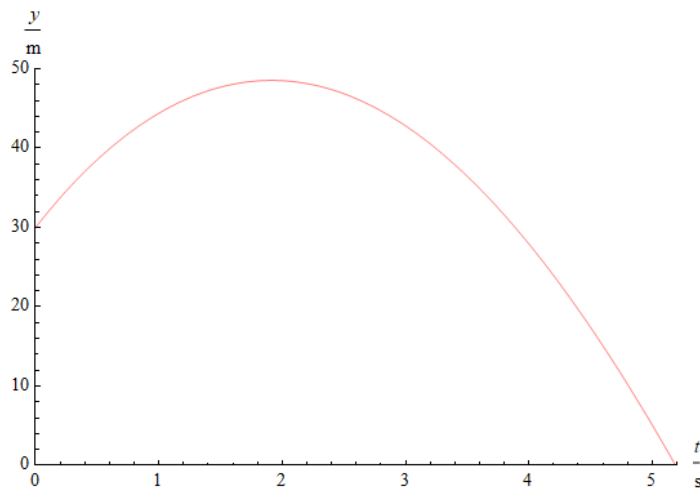
$$t = \sqrt{\frac{1325m}{g}} \left\{ \tan^{-1} \frac{20}{\sqrt{1325mg}} + \cosh^{-1} \left[\exp \left(\frac{6}{265m} \right) \sqrt{1 + \frac{16}{53mg}} \right] \right\} \approx 5.19 \text{ seconds}$$

To conclude, the position and velocity are

$$y(t) = \begin{cases} 1325m \ln \left| \cos \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) \right| + 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}} & \text{if } 0 \leq t \leq t_{\max} \\ -1325m \ln \cosh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) + 30 + 1325m \ln \sqrt{\frac{53mg + 16}{53mg}} & \text{if } t_{\max} \leq t \lesssim 5.19 \end{cases}$$

$$v(t) = \begin{cases} \sqrt{1325mg} \tan \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) & \text{if } 0 \leq t \leq t_{\max} \\ \sqrt{1325mg} \tanh \left(-\sqrt{\frac{g}{1325m}} t + \tan^{-1} \frac{20}{\sqrt{1325mg}} \right) & \text{if } t_{\max} \leq t \lesssim 5.19 \end{cases}$$

Here the position and velocity of the ball subject to air resistance $v^2/1325$ are plotted versus time.



In this next figure, the previous plots are superimposed with the plots of Problem 20 and Problem 21, where there is no air resistance and where there is air resistance $|v|/30$, respectively.

