Problem 23

A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance, which is directed opposite to the velocity, is of magnitude $0.75|v|$ when the parachute is closed and is of magnitude $12|v|$ when the parachute is open, where the velocity v is measured in ft/s.

(a) Find the speed of the skydiver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

(c) What is the limiting velocity v_L after the parachute opens?

(d) Determine how long the sky diver is in the air after the parachute opens.

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Solution

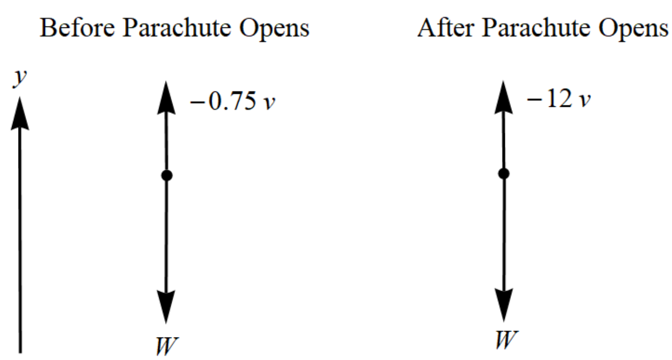
Split up the diver's motion into two parts: one where the parachute is closed and the other where the parachute is open. According to Newton's second law, the equation of motion for the diver is

$$\sum \mathbf{F} = m\mathbf{a}.$$

This is a vector equation, so it actually represents three scalar equations—one for each direction in the chosen coordinate system. Since the diver only moves vertically, only the y-equation is relevant.

$$\sum F_y = ma_y$$

Draw free-body diagrams for the diver before and after the parachute opens.



Minus signs are placed in front of v above because the positive y-axis points upward and the diver is moving downward. The equations of motion can now be written; replace a_y with dv/dt and solve them by using integrating factors.

Before Parachute Opens

$$-0.75v - W = ma_y$$
$$\frac{dv}{dt} + \frac{0.75}{m}v = -\frac{W}{m}$$
$$e^{0.75t/m}\frac{dv}{dt} + \frac{0.75}{m}e^{0.75t/m}v = -\frac{W}{m}e^{0.75t/m}$$
$$\frac{d}{dt}(e^{0.75t/m}v) = -\frac{W}{m}e^{0.75t/m}$$
$$e^{0.75t/m}v = -\frac{W}{0.75}e^{0.75t/m} + C_1$$
$$v(t) = -\frac{W}{0.75} + C_1 e^{-0.75t/m}$$

After Parachute Opens

$$-12v - W = ma_y$$
$$\frac{dv}{dt} + \frac{12}{m}v = -\frac{W}{m}$$
$$e^{12t/m}\frac{dv}{dt} + \frac{12}{m}e^{12t/m}v = -\frac{W}{m}e^{12t/m}$$
$$\frac{d}{dt}(e^{12t/m}v) = -\frac{W}{m}e^{12t/m}$$
$$e^{12t/m}v = -\frac{W}{12}e^{12t/m} + C_2$$
$$v(t) = -\frac{W}{12} + C_2 e^{-12t/m}$$
$$v(t) = -\frac{W}{12} + C_3 e^{-12(t-10)/m}$$

Integrate the velocities to get the positions.

$$y(t) = -\frac{W}{0.75}t - \frac{C_1 m}{0.75}e^{-0.75t/m} + C_4$$

$$y(t) = -\frac{W}{12}t - \frac{C_3 m}{12}e^{-12(t-10)/m} + C_5$$
$$y(t) = -\frac{W}{12}(t-10) - \frac{C_3 m}{12}e^{-12(t-10)/m} + C_6$$

Our task now is to determine the integration constants, C_1, C_3, C_4, and C_6. Assuming that the diver falls from rest, the initial condition $v(0) = 0$ can be used to find C_1.

$$\text{Before:} \quad v(0) = -\frac{W}{0.75} + C_1 = 0 \quad \rightarrow \quad C_1 = \frac{W}{0.75}$$

So then

$$\begin{aligned}\text{Before:} \quad v(t) &= -\frac{W}{0.75} + \frac{W}{0.75}e^{-0.75t/m} \\ &= -\frac{W}{0.75}(1 - e^{-0.75t/m}).\end{aligned}$$

The parachute opens at 10 seconds. The velocity at this time is

$$\text{Before:} \quad v(10) = -\frac{W}{0.75}(1 - e^{-7.5/m}),$$

and it serves as the initial condition that we can use to find C_3.

$$\text{After:} \quad v(10) = -\frac{W}{12} + C_3 = -\frac{W}{0.75}(1 - e^{-7.5/m}) \quad \rightarrow \quad C_3 = \frac{W}{12} - \frac{W}{0.75}(1 - e^{-7.5/m})$$

So then

$$\text{After:} \quad v(t) = -\frac{W}{12} + \left[\frac{W}{12} - \frac{W}{0.75}(1 - e^{-7.5/m})\right]e^{-12(t-10)/m}.$$

With these values of C_1 and C_3, the positions become

$$\text{Before:} \quad y(t) = -\frac{W}{0.75}\left(t + \frac{m}{0.75}e^{-0.75t/m}\right) + C_4$$
$$\text{After:} \quad y(t) = -\frac{W}{12}(t-10) - \left[\frac{W}{12} - \frac{W}{0.75}(1 - e^{-7.5/m})\right]\frac{m}{12}e^{-12(t-10)/m} + C_6.$$

Use the fact that the diver starts at 5000 feet to find C_4.

$$y(0) = -\frac{W}{0.75}\left(\frac{m}{0.75}\right) + C_4 = 5000 \quad \rightarrow \quad C_4 = 5000 + \frac{Wm}{0.5625}$$

So then

$$\text{Before:} \quad y(t) = -\frac{W}{0.75}\left(t + \frac{m}{0.75}e^{-0.75t/m}\right) + 5000 + \frac{Wm}{0.5625}.$$

The position of the diver at 10 seconds (when the parachute is opened) is

$$\text{Before:} \quad y(10) = -\frac{W}{0.75}\left(10 + \frac{m}{0.75}e^{-7.5/m}\right) + 5000 + \frac{Wm}{0.5625}.$$

Because the position of the diver is continuous, this serves as the initial condition to find C_6.

$$\text{After:} \quad y(10) = -\left[\frac{W}{12} - \frac{W}{0.75}(1 - e^{-7.5/m})\right]\frac{m}{12} + C_6 = -\frac{W}{0.75}\left(10 + \frac{m}{0.75}e^{-7.5/m}\right) + 5000 + \frac{Wm}{0.5625}$$

$$C_6 = -\frac{W}{0.75}\left(10 + \frac{m}{0.75}e^{-7.5/m}\right) + 5000 + \frac{Wm}{0.5625} + \left[\frac{W}{12} - \frac{W}{0.75}(1 - e^{-7.5/m})\right]\frac{m}{12}$$

Now the numbers W and m will be plugged in. The weight is $W = 180$ lb, and the mass is

$$m = \frac{W}{g} \approx \frac{180 \text{ lb}}{9.81 \, \frac{\text{m}}{\text{s}^2} \times \frac{3.28 \text{ ft}}{1 \text{ m}}} \approx 5.59 \, \frac{\text{lb} \cdot \text{s}^2}{\text{ft}}.$$

As a result, the constants are

$$\begin{aligned} C_1 &= 240 \\ C_3 &\approx -162.2 \\ C_4 &\approx 6790 \\ C_6 &\approx 3846, \end{aligned}$$

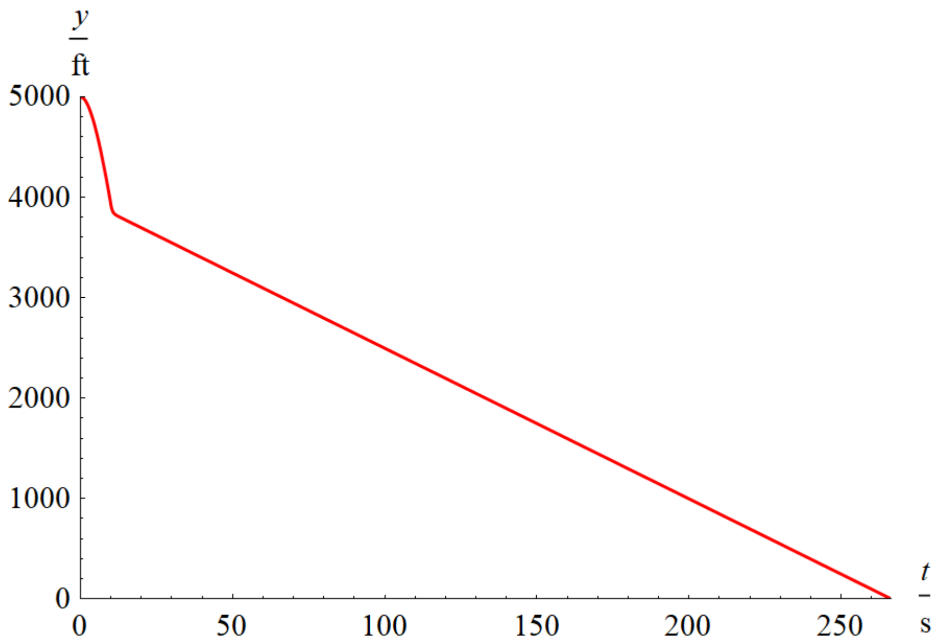
and the position and velocity are

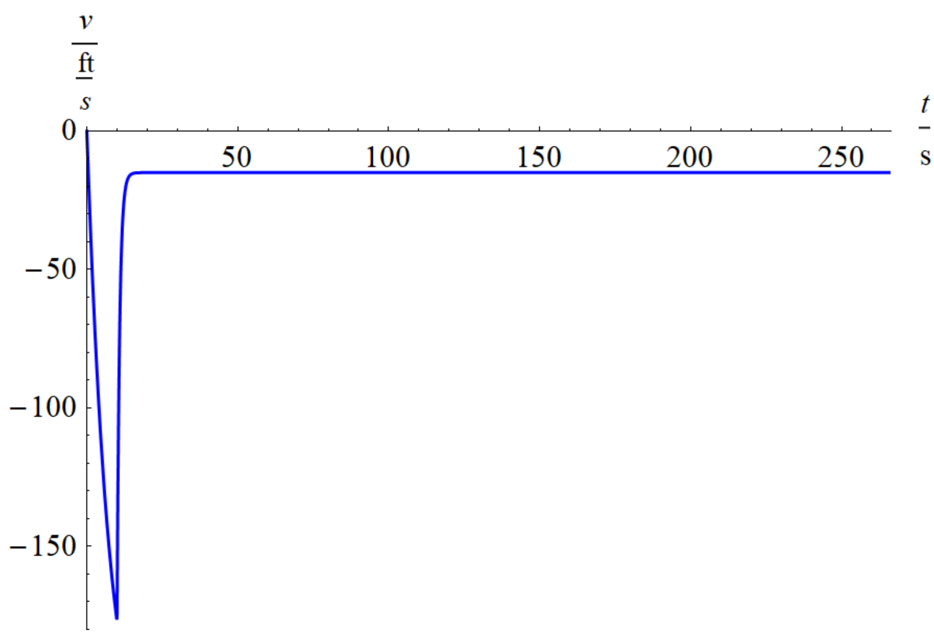
$$y(t) \approx \begin{cases} 6790 - 240(7.459e^{-0.13407t} + t) & 0 \leq t \leq 10 \\ 3846 - 15(t - 10) + 75.61e^{-2.145(t-10)} & 10 \leq t \lesssim 266.4 \end{cases}$$

$$v(t) \approx \begin{cases} -240(1 - e^{-0.13407t}) & 0 \leq t \leq 10 \\ -15 - 162.2e^{-1.2(t-10)} & 10 \leq t \lesssim 266.4 \end{cases}.$$

266.4 seconds is the time it takes for the diver to reach the ground. It was found by graphing $y(t)$ and seeing where the curve crosses the t-axis.

Below are plots of the position and velocity versus time.





The speed of the diver is the magnitude of $v(10)$.

$$|v(10)| \approx 240(1 - e^{-1.3407}) \approx 177.2 \ \frac{\text{ft}}{\text{s}}$$

The diver's position when he opens the parachute is $y(10)$.

$$y(10) \approx 6790 - 240(7.459e^{-1.3407} + 10) \approx 3921.7 \text{ ft}$$

Therefore, he will have travelled a distance of $5000 - 3921.7 \approx 1078.3$ ft after 10 seconds. The limiting velocity after the parachute opens is

$$v_L = \lim_{t\to\infty} v(t) = -\frac{W}{12} = -15 \ \frac{\text{ft}}{\text{s}}.$$

The diver lands at $t \approx 266.4$ seconds and opens the parachute at 10 seconds, so he's in the air for about 256.4 seconds.