Problem 24

A rocket sled having an initial speed of 150 mi/h is slowed by a channel of water. Assume that during the braking process, the acceleration a is given by $a(v) = -\mu v^2$, where v is the velocity and μ is a constant.

(a) As in Example 4 in the text, use the relation $dv/dt = v(dv/dx)$ to write the equation of motion in terms of v and x.

(b) If it requires a distance of 2000 ft to slow the sled to 15 mi/h, determine the value of μ.

(c) Find the time τ required to slow the sled to 15 mi/h.

Solution

Part (a)

The acceleration is given by

$$a = -\mu v^2.$$

Replace a with dv/dt.

$$\frac{dv}{dt} = -\mu v^2$$

Use the chain rule to write v in terms of x:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v.$$

As a result, the previous equation becomes

$$\frac{dv}{dx}v = -\mu v^2.$$

Divide both sides by v^2

$$\frac{\frac{dv}{dx}}{v} = -\mu$$

and then rewrite the left side as a derivative of a logarithm.

$$\frac{d}{dx}\ln|v| = -\mu$$

If we take the positive x-axis to point in the direction the sled is moving, then the absolute value sign can be dropped because v will be positive. Integrate both sides with respect to x.

$$\ln v = -\mu x + C$$

Exponentiate both sides.

$$\begin{aligned} v(x) &= e^{-\mu x + C} \\ &= e^C e^{-\mu x} \end{aligned}$$

Use a new constant A for e^C.

$$v(x) = Ae^{-\mu x}$$

Use the fact that the initial speed is 150 mi/h to determine A.

$$v(0) = A = 150$$

So then

$$v(x) = 150e^{-\mu x}.$$

Part (b)

Now use the fact that the sled requires a distance of 2000 ft to slow to 15 mi/h. Set $v(x) = 15$ and $x = 2000$ and solve for μ.

$$15 = 150e^{-2000\mu}$$

$$e^{-2000\mu} = \frac{1}{10}$$

$$\ln e^{-2000\mu} = \ln\frac{1}{10}$$

$$-2000\mu = -\ln 10$$

$$\mu = \frac{\ln 10}{2000}\,\frac{1}{\text{ft}} \times \frac{5280\text{ ft}}{\text{mi}} = \frac{66}{25}\ln 10\,\frac{1}{\text{mi}} \approx 6.08\,\frac{1}{\text{mi}}$$

Therefore,

$$v(x) = 150\exp\left(-\frac{66\ln 10}{25}x\right),$$

where x is in miles and v is in miles per hour.

Part (c)

Replace $v(x)$ with dx/dt and solve the resulting ODE by separating variables.

$$\frac{dx}{dt} = 150\exp\left(-\frac{66\ln 10}{25}x\right)$$

$$\exp\left(\frac{66\ln 10}{25}x\right)dx = 150\,dt$$

Integrate both sides.

$$\frac{25}{66\ln 10}\exp\left(\frac{66\ln 10}{25}x\right) = 150t + C_2$$

Apply the initial condition $x(0) = 0$ to determine C_2.

$$\frac{25}{66\ln 10} = C_2$$

The previous equation is then

$$\frac{25}{66\ln 10}\exp\left(\frac{66\ln 10}{25}x\right) = 150t + \frac{25}{66\ln 10}.$$

Solve for t.

$$150t = \frac{25}{66\ln 10}\left[\exp\left(\frac{66\ln 10}{25}x\right) - 1\right]$$

Divide both sides by 150.

$$t = \frac{1}{396 \ln 10}\left[\exp\left(\frac{66 \ln 10}{25}x\right) - 1\right].$$

Therefore, the time τ it takes for the sled to go

$$x = 2000 \text{ ft} \times \frac{1 \text{ mi}}{5280 \text{ ft}} = \frac{25}{66} \text{ mi}$$

is

$$\tau = \frac{1}{44 \ln 10} \text{ h} \times \frac{3600 \text{ seconds}}{1 \text{ hour}} \approx 35.53 \text{ seconds}.$$