Problem 26

A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

(a) Find the velocity $v(t)$ of the body at any time.

(b) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$—that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?

(c) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$—that is, as the mass approaches zero.

Solution

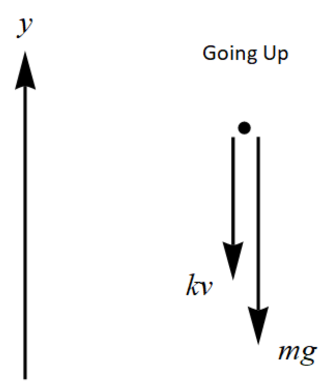
According to Newton's second law, the equation of motion for the mass is

$$\sum \mathbf{F} = m\mathbf{a}.$$

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the y-direction is relevant.

$$\sum F_y = ma_y$$

Draw the free-body diagram for the mass as it's travelling upward.



Now the equation of motion can be written.

$$-kv - mg = ma_y$$

Replace a_y with dv/dt, bring kv to the other side, and divide both sides by m.

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{k}{m}\,ds\right) = e^{kt/m}$$

Proceed with the multiplication.

$$e^{kt/m}\frac{dv}{dt} + \frac{k}{m}e^{kt/m}v = -ge^{kt/m}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}(e^{kt/m}v) = -ge^{kt/m}$$

Integrate both sides with respect to t.

$$e^{kt/m}v = -\frac{mg}{k}e^{kt/m} + C_1$$

Divide both sides by $e^{kt/m}$.

$$v(t) = -\frac{mg}{k} + C_1 e^{-kt/m}$$

Apply the initial condition $v(0) = v_0$ to determine C_1.

$$v(0) = -\frac{mg}{k} + C_1 = v_0 \quad \rightarrow \quad C_1 = v_0 + \frac{mg}{k}$$

Therefore,

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

The Taylor series expansion of e^{-x} about $x = 0$ is

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots,$$

so in the limit as $k \rightarrow 0$, the exponential function can be replaced by this expansion.

$$\begin{aligned}
\lim_{k\to 0} v(t) &= \lim_{k\to 0}\left\{-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)\left[1 - \left(\frac{kt}{m}\right) + \frac{1}{2}\left(\frac{kt}{m}\right)^2 - \frac{1}{6}\left(\frac{kt}{m}\right)^3 + \cdots\right]\right\} \\
&= \lim_{k\to 0}\left\{-\frac{mg}{k} + v_0 + \frac{mg}{k} - v_0\frac{kt}{m} - \frac{mg}{k}\frac{kt}{m} + \left(v_0 + \frac{mg}{k}\right)\left[\frac{1}{2}\left(\frac{kt}{m}\right)^2 - \frac{1}{6}\left(\frac{kt}{m}\right)^3 + \cdots\right]\right\} \\
&= \lim_{k\to 0}\left\{v_0 - v_0\frac{kt}{m} - gt + k(kv_0 + mg)\left[\frac{1}{2}\left(\frac{t}{m}\right)^2 - \frac{k}{6}\left(\frac{t}{m}\right)^3 + \cdots\right]\right\} \\
&= v_0 - gt
\end{aligned}$$

This result agrees with the velocity in a vacuum $v = v_0 + at = v_0 - gt$. Now take the limit of $v(t)$ as $m \rightarrow 0$.

$$\begin{aligned}
\lim_{m\to 0} v(t) &= \lim_{m\to 0}\left[-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}\right] \\
&= \lim_{m\to 0} v_0 e^{-kt/m} \\
&= v_0 e^{-\infty} \\
&= 0
\end{aligned}$$