Problem 27

A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law, $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.[7]

(a) Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

(b) In 1910 R. A. Millikan[8] studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary ($v = 0$) and that w and B are as given above. Find an expression for e. Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

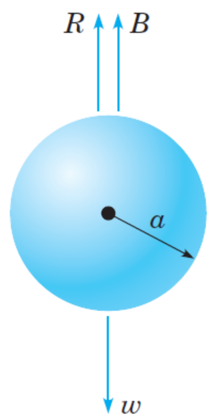


FIGURE 2.3.5 A body falling in a dense fluid.

Solution

According to Newton's second law, the equation of motion for the mass is

$$\sum \mathbf{F} = m\mathbf{a}.$$

This is a vector equation; it consists of three scalar equations—one for each direction in the chosen coordinate system. Since the mass is projected vertically upward, only the equation in the y-direction is relevant.

$$\sum F_y = ma_y$$

[7]Sir George Gabriel Stokes (1819–1903) was born in Ireland but for most of his life was at Cambridge University, first as a student and later as a professor. Stokes was one of the foremost applied mathematicians of the nineteenth century, best known for his work in fluid dynamics and the wave theory of light. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series.

[8]Robert A. Millikan (1868–1953) was educated at Oberlin College and Columbia University. Later he was a professor at the University of Chicago and California Institute of Technology. His determination of the charge on an electron was published in 1910. For this work, and for other studies of the photoelectric effect, he was awarded the Nobel Prize for Physics in 1923.

Part (a)

Use the free-body diagram drawn in Figure 2.3.5 to write the equation of motion. (Take the positive y-axis to point downward in the direction of the motion.)

$$w - R - B = ma_y \tag{1}$$

Now we'll write expressions for each of the variables. w is the weight of the solid ball; multiply its volume by its density by g to get force.

$$w = \frac{4}{3}\pi a^3 \times \rho \times g$$

R is the resistive force given by Stokes's law.

$$R = 6\pi\mu a v$$

B is the buoyant force; multiply the volume of the ball by the density of the medium by g to get force.

$$B = \frac{4}{3}\pi a^3 \times \rho' \times g$$

m is the mass of the ball; multiply its volume by its density to get the mass.

$$m = \frac{4}{3}\pi a^3 \times \rho$$

Finally, replace a_y with dv/dt. As a result, equation (1) becomes

$$\frac{4}{3}\pi a^3 \rho g - 6\pi\mu a v - \frac{4}{3}\pi a^3 \rho' g = \frac{4}{3}\pi a^3 \rho \frac{dv}{dt}.$$

Simplify both sides of the equation.

$$\frac{4}{3}\pi a^3 (\rho - \rho') g - 6\pi\mu a v = \frac{4}{3}\pi a^3 \rho \frac{dv}{dt}$$

$$\left(1 - \frac{\rho'}{\rho}\right) g - \frac{9\mu}{2a^2\rho} v = \frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{9\mu}{2a^2\rho} v = \left(1 - \frac{\rho'}{\rho}\right) g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{9\mu}{2a^2\rho}\, ds\right) = e^{9\mu t/(2a^2\rho)}$$

Proceed with the multiplication.

$$e^{9\mu t/(2a^2\rho)} \frac{dv}{dt} + \frac{9\mu}{2a^2\rho} e^{9\mu t/(2a^2\rho)} v = \left(1 - \frac{\rho'}{\rho}\right) g e^{9\mu t/(2a^2\rho)}$$

The left side can be written as $d/dt(Iv)$ by the product rule.

$$\frac{d}{dt}[e^{9\mu t/(2a^2\rho)} v] = \left(1 - \frac{\rho'}{\rho}\right) g e^{9\mu t/(2a^2\rho)}$$

Integrate both sides with respect to t.

$$e^{9\mu t/(2a^2\rho)}v = \frac{2a^2\rho}{9\mu}\left(1 - \frac{\rho'}{\rho}\right) g e^{9\mu t/(2a^2\rho)} + C_1$$

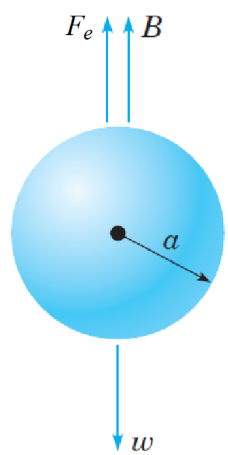
Divide both sides by $e^{9\mu t/(2a^2\rho)}$ and distribute ρ.

$$v(t) = \frac{2a^2}{9\mu}(\rho - \rho')g + C_1 e^{-9\mu t/(2a^2\rho)}$$

In the limit as $t \to \infty$, the exponential function decays to zero, and only the first term remains.

$$\lim_{t\to\infty} v(t) = \frac{2a^2}{9\mu}(\rho - \rho')g$$

Part (b)



Since the oil droplet is stationary, the sum of the forces must be equal to zero.

$$w - F_e - B = 0$$

$$w = \frac{4}{3}\pi a^3 \rho g$$
$$F_e = eE$$
$$B = \frac{4}{3}\pi a^3 \rho' g$$

$$\frac{4}{3}\pi a^3 \rho g - eE - \frac{4}{3}\pi a^3 \rho' g = 0$$
$$eE = \frac{4}{3}\pi a^3(\rho - \rho')g$$

Therefore, the electron charge is

$$e = \frac{4}{3E}\pi a^3(\rho - \rho')g.$$