Problem 29

Suppose that a rocket is launched straight up from the surface of the earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of the earth. Neglect air resistance.

(a) Find an expression for the velocity v in terms of the distance x from the surface of the earth.

(b) Find the time required for the rocket to go 240,000 mi (the approximate distance from the earth to the moon). Assume that $R = 4000$ mi.

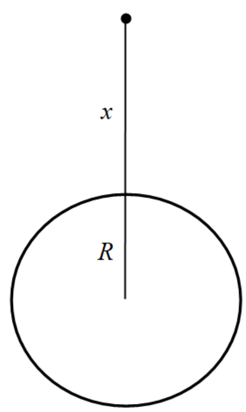
Solution

Part (a)

According to Newton's law of gravitation, the force between two bodies is inversely proportional to the square of the distance between them.

$$F \propto -\frac{1}{r^2}$$

The minus sign indicates that the force is attractive rather than repulsive. For an extended body such as the earth, the mass can be treated as if it were concentrated at the center of mass. Let R represent the radius of Earth and let x represent the distance the rocket is above the earth's surface.



Then $r = R + x$.

$$F \propto -\frac{1}{(R+x)^2}$$

This proportionality can be changed to an equation if a constant k is introduced on the right side.

$$F = -\frac{k}{(R+x)^2}$$

At the earth's surface, the gravitational force is $-mg$; use this fact to determine k.

$$-mg = -\frac{k}{R^2} \quad \rightarrow \quad k = mgR^2$$

As a result, the force acting on the rocket is

$$F(x) = -\frac{mgR^2}{(R+x)^2}.$$

According to Newton's second law, force is mass times acceleration.

$$ma = -\frac{mgR^2}{(R+x)^2}$$

Divide both sides by m

$$a(x) = -\frac{gR^2}{(R+x)^2}$$

and replace a with dv/dt.

$$\frac{dv}{dt} = -\frac{gR^2}{(R+x)^2}$$

By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v,$$

so the previous equation becomes

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE can be solved by separating variables.

$$v\,dv = -\frac{gR^2}{(R+x)^2}\,dx$$

Integrate both sides.

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + C$$

Multiply both sides by 2.

$$v^2 = \frac{2gR^2}{R+x} + 2C$$

Take the square root both sides and use a new constant C_1 for $2C$.

$$v(x) = \pm\sqrt{\frac{2gR^2}{R+x} + C_1}$$

Since the rocket is being launched away from the earth, we choose the plus sign.

$$v(x) = \sqrt{\frac{2gR^2}{R+x} + C_1}$$

Now use the initial velocity to determine C_1. The velocity on the surface is $\sqrt{2gR}$, so

$$\sqrt{2gR} = \sqrt{\frac{2gR^2}{R} + C_1}.$$

Squaring both sides, we find that $0 = C_1$. Therefore,

$$v(x) = \sqrt{\frac{2gR^2}{R+x}} = R\sqrt{\frac{2g}{R+x}}.$$

Part (b)

Replace v with dx/dt to get an ODE for the position.

$$\frac{dx}{dt} = R\sqrt{\frac{2g}{R+x}}$$

Solve this ODE by separating variables.

$$\sqrt{x+R}\,dx = R\sqrt{2g}\,dt$$

Integrate both sides.

$$\frac{2}{3}(x+R)^{3/2} = R\sqrt{2g}t + C_2$$

Since the rocket is on the ground initially, $x(0) = 0$. Use this initial condition to determine C_2.

$$\frac{2}{3}R^{3/2} = C_2$$

Consequently,

$$\frac{2}{3}(x+R)^{3/2} = R\sqrt{2g}t + \frac{2}{3}R^{3/2}.$$

Solve for t.

$$R\sqrt{2g}t = \frac{2}{3}(x+R)^{3/2} - \frac{2}{3}R^{3/2}$$

$$t = \frac{2}{3R\sqrt{2g}}[(x+R)^{3/2} - R^{3/2}]$$

Now plug in $x = 240\,000$ mi and $R = 4000$ mi and

$$g = 9.81\ \frac{\text{m}}{\text{s}^2} \times \frac{3.28\text{ ft}}{1\text{ m}} \times \frac{1\text{ mi}}{5280\text{ ft}} \times \left(\frac{3600\text{ s}}{1\text{ hr}}\right)^2 \approx 78\,979\ \frac{\text{mi}}{\text{hr}^2}$$

and evaluate t.

$$t \approx 50\text{ hr}$$