

Problem 31

A more realistic model (than that in Problem 30) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$dv/dt = -rv, \quad dw/dt = -g - rw,$$

where r is the coefficient of resistance.

- Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A .
- Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.
- Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A . How do the trajectories differ from those in [Problem 31](#) with $r = 0$?
- Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 30(f).

[**TYP0**: r , h , and u should be written as $r = 1/5 \text{ s}^{-1}$, $u = 125 \text{ ft/s}$, and $h = 3 \text{ ft}$. Also, [Problem 31](#) needs to be changed to [Problem 30](#).]

Solution

Part (a)

Both the ODE for v and the ODE for w can be solved by multiplying both sides by an integrating factor.

$$\begin{aligned} \frac{dv}{dt} &= -rv \\ \frac{dv}{dt} + rv &= 0 \end{aligned}$$

$$I_1 = \exp\left(\int^t r \, ds\right) = e^{rt}$$

$$e^{rt} \frac{dv}{dt} + r e^{rt} v = 0$$

$$\frac{d}{dt}(e^{rt}v) = 0$$

$$e^{rt}v = C_1$$

$$v(t) = C_1 e^{-rt}$$

$$\begin{aligned} \frac{dw}{dt} &= -g - rw \\ \frac{dw}{dt} + rw &= -g \end{aligned}$$

$$I_2 = \exp\left(\int^t r \, ds\right) = e^{rt}$$

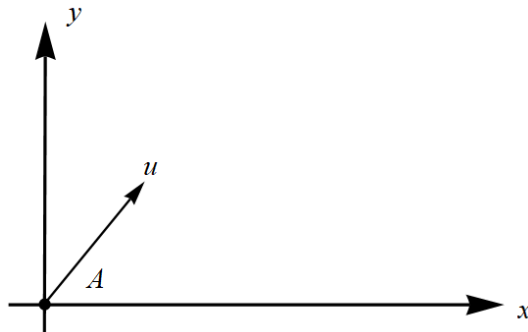
$$e^{rt} \frac{dw}{dt} + r e^{rt} w = -g e^{rt}$$

$$\frac{d}{dt}(e^{rt}w) = -g e^{rt}$$

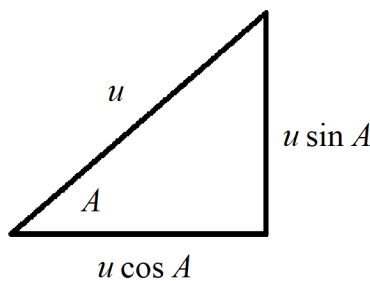
$$e^{rt}w = -\frac{g}{r} e^{rt} + C_2$$

$$w(t) = -\frac{g}{r} + C_2 e^{-rt}$$

A schematic of the baseball being launched with initial speed u at an angle A is shown below.



Decompose the initial velocity vector into its components along the x - and y -axes.



Use these components to determine C_1 and C_2 .

$$v(0) = C_1 = u \cos A \qquad w(0) = -\frac{g}{r} + C_2 = u \sin A \quad \rightarrow \quad C_2 = \frac{g}{r} + u \sin A$$

Therefore,

$$v(t) = ue^{-rt} \cos A \qquad w(t) = -\frac{g}{r} + \left(\frac{g}{r} + u \sin A\right) e^{-rt}.$$

Part (b)

Replace v with dx/dt and replace w with dy/dt .

$$\frac{dx}{dt} = ue^{-rt} \cos A \qquad \frac{dy}{dt} = -\frac{g}{r} + \left(\frac{g}{r} + u \sin A\right) e^{-rt}$$

Integrate the velocities to get the positions.

$$x(t) = -\frac{u}{r}e^{-rt} \cos A + C_3 \qquad y(t) = -\frac{g}{r}t - \frac{1}{r} \left(\frac{g}{r} + u \sin A\right) e^{-rt} + C_4$$

Apply the initial conditions to determine C_3 and C_4 .

$$\begin{aligned} x(0) &= -\frac{u}{r} \cos A + C_3 = 0 & y(0) &= -\frac{1}{r} \left(\frac{g}{r} + u \sin A\right) + C_4 = h \\ C_3 &= \frac{u}{r} \cos A & C_4 &= h + \frac{1}{r} \left(\frac{g}{r} + u \sin A\right) \end{aligned}$$

Therefore,

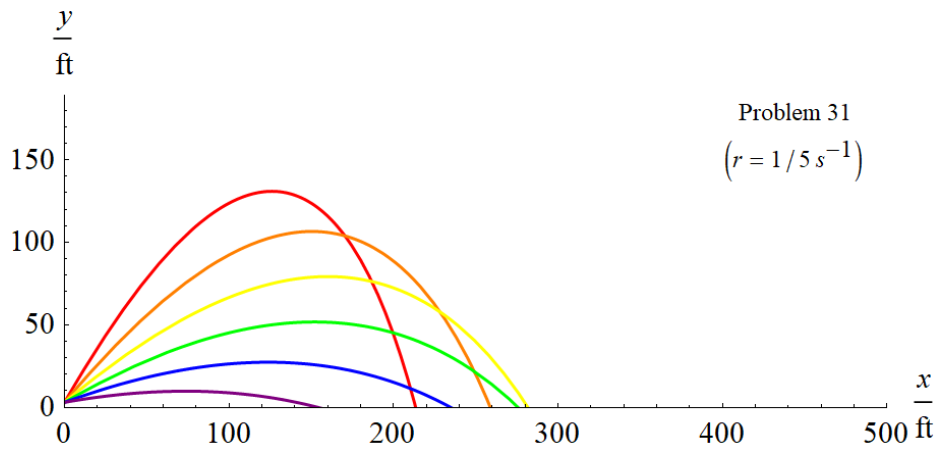
$$\begin{aligned}
 x(t) &= -\frac{u}{r}e^{-rt} \cos A + \frac{u}{r} \cos A & y(t) &= -\frac{g}{r}t - \frac{1}{r} \left(\frac{g}{r} + u \sin A \right) e^{-rt} + h + \frac{1}{r} \left(\frac{g}{r} + u \sin A \right) \\
 x(t) &= \frac{u}{r}(1 - e^{-rt}) \cos A & y(t) &= h - \frac{g}{r}t + \frac{1}{r} \left(\frac{g}{r} + u \sin A \right) (1 - e^{-rt}).
 \end{aligned}$$

Part (c)

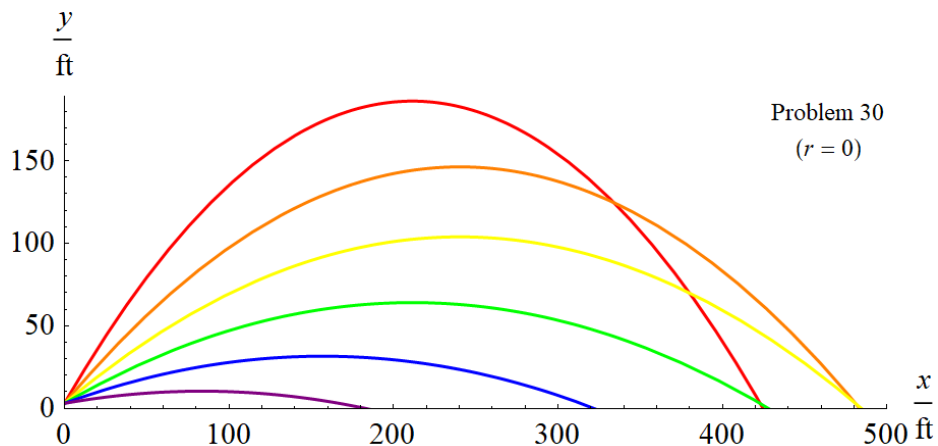
Plug in $r = 1/5 \text{ s}^{-1}$, $u = 125 \text{ ft/s}$, $h = 3 \text{ ft}$, and $g = 32 \text{ ft/s}^2$.

$$\begin{aligned}
 x(t) &= 625(1 - e^{-t/5}) \cos A \\
 y(t) &= 3 - 160t + 5(160 + 125 \sin A)(1 - e^{-t/5})
 \end{aligned}$$

Below are graphs for $A = 10^\circ$, $A = 20^\circ$, $A = 30^\circ$, $A = 40^\circ$, $A = 50^\circ$, and $A = 60^\circ$ in purple, blue, green, yellow, orange, and red, respectively.



The results of Problem 30 (no air resistance) are shown below for comparison.



Part (d)

The results of part (b) tell us the position of the baseball at any time.

$$x(t) = \frac{u}{r}(1 - e^{-rt}) \cos A$$

$$y(t) = h - \frac{g}{r}t + \frac{1}{r} \left(\frac{g}{r} + u \sin A \right) (1 - e^{-rt})$$

Set $x(t) = 350$ ft, $y(t) = 10$ ft, $r = 1/5$ s⁻¹, $h = 3$ ft, and $g = 32$ ft/s².

$$350 = 5u(1 - e^{-t/5}) \cos A$$

$$10 = 3 - 160t + 5(160 + u \sin A)(1 - e^{-t/5})$$

Solve the first equation for $e^{-t/5}$ and t

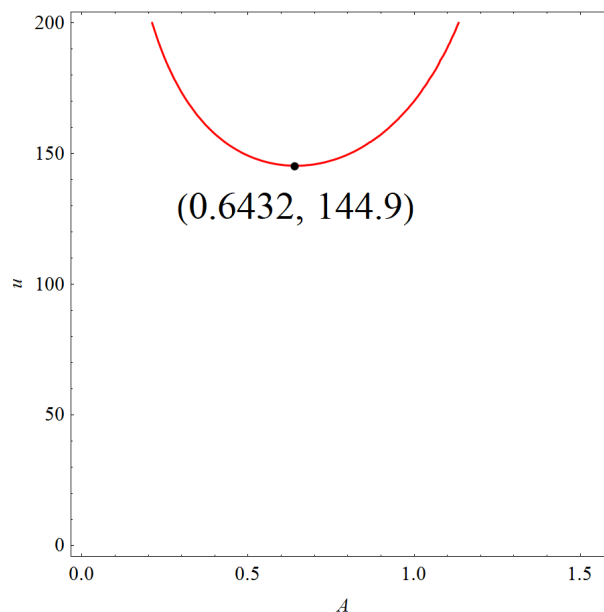
$$350 = 5u(1 - e^{-t/5}) \cos A \quad \rightarrow \quad \begin{cases} e^{-t/5} = 1 - \frac{350}{5u \cos A} \\ t = -5 \ln \left(1 - \frac{350}{5u \cos A} \right) \end{cases}$$

and then plug these formulas into the second equation to eliminate t .

$$10 = 3 + 800 \ln \left(1 - \frac{350}{5u \cos A} \right) + 5(160 + u \sin A) \left(\frac{350}{5u \cos A} \right)$$

$$7 = 800 \ln \left(1 - \frac{70}{u \cos A} \right) + \frac{56000}{u \cos A} + 350 \tan A$$

This is an implicit function of u . Below is a plot of u versus A , and the minimum is marked.



Therefore, the minimum speed needed to get to (350 ft, 10 ft) is about 145 ft/s, and the corresponding angle is about 0.64 radians, or 37°.