Problem 4

A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Solution

Let t represent the time in minutes, let $V = V(t)$ represent the volume in gallons, and let $m = m(t)$ represent the mass of salt in pounds. The tank initially contains 200 gal of water and 100 lb of salt.

$$\begin{aligned} V(0) &= 200 \text{ gal} \\ m(0) &= 100 \text{ lb} \end{aligned}$$

According to the law of conservation of mass, mass is neither created nor destroyed. If solution flows into a tank at some rate, then it must flow out at the same rate; otherwise, it will accumulate in the tank.

$$\text{rate of accumulation} = \text{rate flowing in} - \text{rate flowing out}$$

Apply this law to the volume, noting that dV/dt is the rate that volume increases with respect to time.

$$\begin{aligned} \frac{dV}{dt} &= 3\ \frac{\text{gal}}{\text{min}} - 2\ \frac{\text{gal}}{\text{min}} \\ &= 1 \end{aligned}$$

Integrate both sides with respect to t.

$$V(t) = t + C_1$$

Use the initial condition for V to determine C_1.

$$V(0) = C_1 = 200 \text{ gal}$$

So the volume is

$$V(t) = t + 200 \text{ gal},$$

which means the tank starts to overflow at $t = 300$ min. Now apply the law to the mass, noting that dm/dt is the rate that the mass of salt increases with respect to time. To obtain the rate of mass flow, multiply the concentration by the volume flow rate. Assuming the solution is well-stirred, the concentration flowing out is $m(t)/V(t)$.

$$\begin{aligned} \frac{dm}{dt} &= \left(3\ \frac{\text{gal}}{\text{min}}\right)\left(1\ \frac{\text{lb}}{\text{gal}}\right) - \left(2\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{m(t)}{V(t)}\right) \\ &= 3 - \frac{2m}{t+200} \end{aligned}$$

Bring $2m/(t+200)$ to the left side.

$$\frac{dm}{dt} + \frac{2m}{t+200} = 3$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{2}{t+200}\,ds\right) = e^{2\ln(t+200)} = e^{\ln(t+200)^2} = (t+200)^2$$

Proceed with the multiplication.

$$(t+200)^2\frac{dm}{dt} + 2(t+200)m = 3(t+200)^2$$

The left side can be written as $d/dt(Im)$ by the product rule.

$$\frac{d}{dt}[(t+200)^2 m] = 3t^2 + 1200t + 120\,000$$

Integrate both sides with respect to t.

$$(t+200)^2 m = t^3 + 600t^2 + 120\,000t + C_2$$

Divide both sides by $(t+200)^2$.

$$m(t) = \frac{t^3 + 600t^2 + 120\,000t + C_2}{(t+200)^2}$$

Use the initial condition for m to determine C_2.

$$m(0) = \frac{C_2}{200^2} = 100 \quad \rightarrow \quad C_2 = 4\,000\,000$$

So then the mass of salt is

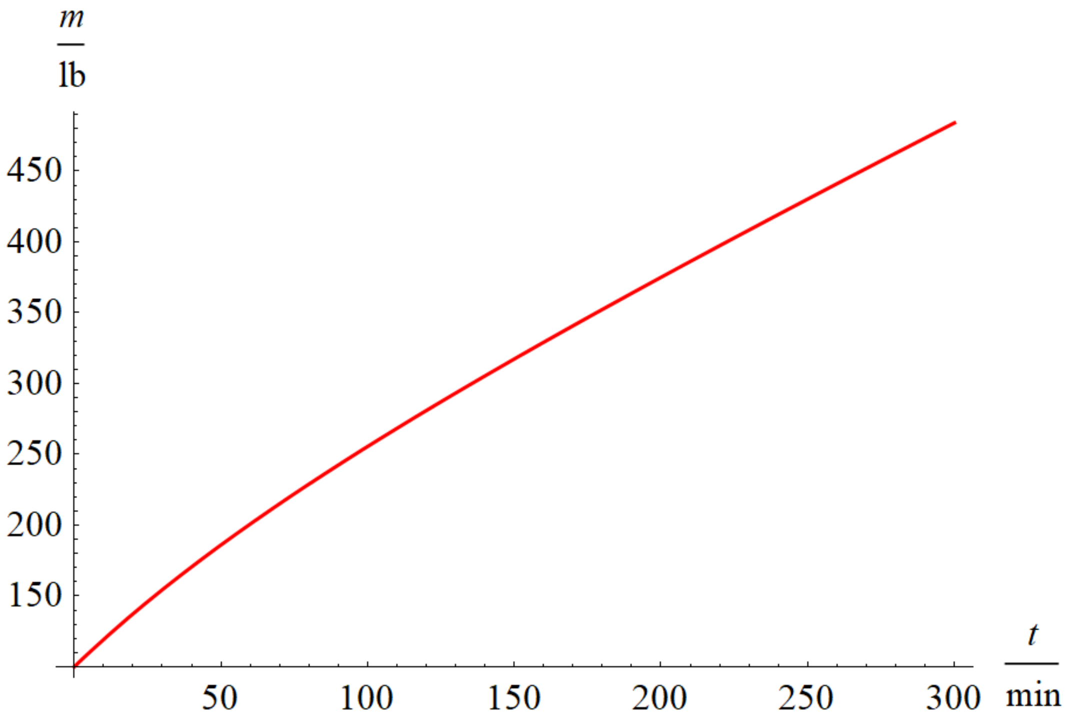
$$m(t) = \frac{t^3 + 600t^2 + 120\,000t + 4\,000\,000}{(t+200)^2}\text{ lb}, \quad 0 \le t \le 300$$

for the first 300 minutes. The concentration of salt in the tank at the point of overflow is

$$\frac{m(300)}{V(300)} = \frac{121}{125}\,\frac{\text{lb}}{\text{gal}} \approx 0.968\,\frac{\text{lb}}{\text{gal}}.$$

If the tank had infinite capacity, then the limiting concentration as $t \to \infty$ would be

$$\begin{aligned}\lim_{t\to\infty}\frac{m(t)}{V(t)} = \lim_{t\to\infty}\frac{\frac{t^3+600t^2+120\,000t+4\,000\,000}{(t+200)^2}}{t+200} &= \lim_{t\to\infty}\frac{t^3 + 600t^2 + 120\,000t + 4\,000\,000}{(t+200)^3} \\ &= \lim_{t\to\infty}\frac{1 + \frac{600}{t} + \frac{120\,000}{t^2} + \frac{4\,000\,000}{t^3}}{\left(1+\frac{200}{t}\right)^3} \\ &= 1\,\frac{\text{lb}}{\text{gal}}.\end{aligned}$$



The percent difference between the concentration at the point of overflow and the theoretical limiting concentration is

$$\frac{\frac{121}{125} - 1}{1} \times 100\% = -3.2\%,$$

which means that at overflow the concentration is 3.2% less than the theoretical limit.