Problem 7

Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

(a) Find the time T required for the original sum to double in value as a function of r.

(b) Determine T if $r = 7\%$.

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

Solution

If the compounding is continuous, then the rate dS/dt that the sum of money grows with respect to time is equal to the annual interest r times S.

$$\frac{dS}{dt} = rS$$

Divide both sides by S.

$$\frac{\frac{dS}{dt}}{S} = r$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dt}\ln S = r$$

Integrate both sides with respect to t.

$$\ln S = rt + C$$

Exponentiate both sides.

$$\begin{aligned} S(t) &= e^{rt+C} \\ &= e^C e^{rt} \end{aligned}$$

Use a new constant A for e^C.

$$S(t) = Ae^{rt}$$

Assuming that the initial investment is $S(0) = S_0$, the constant A evaluates to

$$S(0) = A = S_0.$$

So then the amount of money available at any time t (in years) is

$$S(t) = S_0 e^{rt}.$$

Part (a)

Set $S = 2S_0$ and solve for $t = T$ to determine how long it takes for the initial investment to double in value.

$$2S_0 = S_0 e^{rT}$$
$$e^{rT} = 2$$
$$\ln e^{rT} = \ln 2$$
$$rT = \ln 2$$

Therefore,

$$T = \frac{\ln 2}{r}.$$

Part (b)

If $r = 7\% = 0.07$, then

$$T = \frac{\ln 2}{0.07} \approx 9.90 \text{ years.}$$

Part (c)

Set $T = 8$ and solve for r in the result of part (a).

$$8 = \frac{\ln 2}{r}$$
$$r = \frac{\ln 2}{8} \approx 0.0866 = 8.66\%$$