Problem 8

A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time t.

(b) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

(c) If $k = \$2000$/year, determine the return rate r that must be obtained to have \$1 million available in 40 years.

Solution

Part (a)

The young man's capital $S(t)$ grows in time due to two factors, the compound interest and his continuous investments. The rate of growth for compounding is rS, and the rate of growth due to the continuous investments is k.

$$\frac{dS}{dt} = rS + k$$

Bring rS to the left side.

$$\frac{dS}{dt} - rS = k$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t (-r)\, ds\right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt}\frac{dS}{dt} - re^{-rt}S = ke^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt}S) = ke^{-rt}$$

Integrate both sides with respect to t.

$$e^{-rt}S = -\frac{k}{r}e^{-rt} + C$$

Multiply both sides by e^{rt}.

$$S(t) = -\frac{k}{r} + Ce^{rt}$$

Apply the initial condition $S(0) = 0$ to determine C.

$$S(0) = -\frac{k}{r} + C = 0 \quad \rightarrow \quad C = \frac{k}{r}$$

Therefore, the young man's capital after t years is

$$\begin{aligned} S(t) &= -\frac{k}{r} + \frac{k}{r}e^{rt} \\ &= \frac{k}{r}(e^{rt} - 1). \end{aligned}$$

Part (b)

Set $r = 0.075$, $S(t) = 1\,000\,000$, $t = 40$, and solve the resulting equation for k.

$$1\,000\,000 = \frac{k}{0.075}(e^{0.075 \cdot 40} - 1)$$

$$k = \frac{0.075 \cdot 1\,000\,000}{e^{0.075 \cdot 40} - 1} \approx 3930 \ \frac{\text{dollars}}{\text{year}}$$

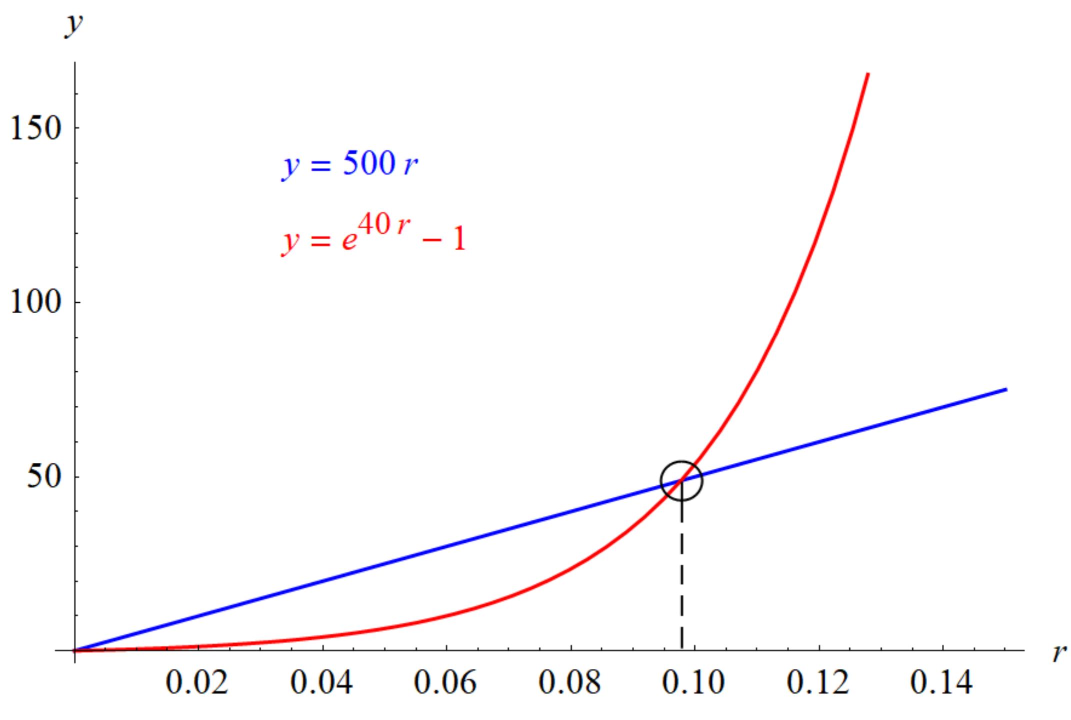
Part (c)

Set $k = 2000$, $S(t) = 1\,000\,000$, $t = 40$, and solve the resulting equation for r.

$$1\,000\,000 = \frac{2000}{r}(e^{40r} - 1)$$

$$500r = e^{40r} - 1$$

Plot $y = 500r$ and $y = e^{40r} - 1$ on the same axis and find where the two curves intersect.



We see that $r \approx 0.098 = 9.8\%$. This is how high the annual interest rate has to be to have a million dollars in 40 years by only continuously investing \$2000 annually.