Problem 9

A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

Solution

The amount of money $S(t)$ that the graduate has to pay changes in time due to two factors, the compound interest and his continuous payments. The rate of growth for compounding is rS, and the rate of decay due to the continuous payments is k.

$$\frac{dS}{dt} = rS - k$$

Bring rS to the left side.

$$\frac{dS}{dt} - rS = -k$$

This is a linear first-order inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t (-r)\,ds\right) = e^{-rt}$$

Proceed with the multiplication.

$$e^{-rt}\frac{dS}{dt} - re^{-rt}S = -ke^{-rt}$$

The left side can be written as $d/dt(IS)$ by the product rule.

$$\frac{d}{dt}(e^{-rt}S) = -ke^{-rt}$$

Integrate both sides with respect to t.

$$e^{-rt}S = \frac{k}{r}e^{-rt} + C$$

Multiply both sides by e^{rt}.

$$S(t) = \frac{k}{r} + Ce^{rt}$$

Apply the initial condition $S(0) = 8000$ to determine C.

$$S(0) = \frac{k}{r} + C = 8000 \quad \rightarrow \quad C = 8000 - \frac{k}{r}$$

Therefore, the amount of money the graduate has to pay after t years is

$$\begin{aligned} S(t) &= \frac{k}{r} + \left(8000 - \frac{k}{r}\right)e^{rt} \\ &= \frac{k}{r}(1 - e^{rt}) + 8000e^{rt}. \end{aligned}$$

Set $S(t) = 0$, $r = 10\% = 0.1$, $t = 3$, and solve the resulting equation for k.

$$0 = \frac{k}{0.1}(1 - e^{0.1\cdot 3}) + 8000e^{0.1\cdot 3}$$

$$k = \frac{0.1 \cdot 8000e^{0.3}}{e^{0.3} - 1} \approx 3086.64 \,\frac{\text{dollars}}{\text{year}}$$

This is the payment rate that is necessary to pay off the loan in three years. The total amount paid in three years is then approximately

$$3 \times 3086.64 \approx \$9259.91.$$

Therefore, the amount paid in interest is approximately

$$\$9259.91 - \$8000 \approx \$1259.91.$$