Problem 21

Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.

(a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

(b) Is there a solution that passes through the point $(2, 1)$? If so, find it.

(c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

Solution

Suppose first that $y \neq 0$. Divide both sides of the ODE by $y^{1/3}$.

$$\frac{y'}{y^{1/3}} = 1$$

The left side can be written as $d/dt[(3/2)y^{2/3}]$ by using the chain rule.

$$\frac{d}{dt}\left(\frac{3}{2}y^{2/3}\right) = 1$$

Integrate both sides with respect to t to obtain the general solution.

$$\frac{3}{2}y^{2/3} = t + C \tag{1}$$

Apply the initial condition $y(0) = 0$ to determine C.

$$0 = C$$

So then the previous equation becomes

$$\frac{3}{2}y^{2/3} = t$$

$$y^{2/3} = \frac{2t}{3}$$

$$y^2 = \left(\frac{2t}{3}\right)^3.$$

Therefore, two nonzero solutions that pass through $(0, 0)$ are

$$y(t) = \pm\left(\frac{2t}{3}\right)^{3/2}.$$

By inspection, we see that $y(t) = 0$ also satisfies the initial value problem. The set of values at $t = 2$ are

$$y(2) = \left\{0, \pm\left(\frac{4}{3}\right)^{3/2}\right\} \approx \{0, \pm 1.5396\}.$$

Part (a)

Rather than $y(0) = 0$, apply the initial condition $y(1) = 1$ to find C instead.

$$\frac{3}{2} = 1 + C \quad \rightarrow \quad C = \frac{1}{2}$$

So then equation (1) becomes

$$\frac{3}{2}y^{2/3} = t + \frac{1}{2}$$
$$y^{2/3} = \frac{2}{3}t + \frac{1}{3}$$
$$y^2 = \left(\frac{2}{3}t + \frac{1}{3}\right)^3$$
$$y(t) = \pm\left(\frac{2}{3}t + \frac{1}{3}\right)^{3/2}.$$

Therefore, the solution that passes through $(1, 1)$ is

$$y(t) = \left(\frac{2}{3}t + \frac{1}{3}\right)^{3/2}.$$

Part (b)

Rather than $y(0) = 0$, apply the initial condition $y(2) = 1$ to find C instead.

$$\frac{3}{2} = 2 + C \quad \rightarrow \quad C = -\frac{1}{2}$$

So then equation (1) becomes

$$\frac{3}{2}y^{2/3} = t - \frac{1}{2}$$
$$y^{2/3} = \frac{2}{3}t - \frac{1}{3}$$
$$y^2 = \left(\frac{2}{3}t - \frac{1}{3}\right)^3$$
$$y(t) = \pm\left(\frac{2}{3}t - \frac{1}{3}\right)^{3/2}.$$

Therefore, the solution that passes through $(2, 1)$ is

$$y(t) = \left(\frac{2}{3}t - \frac{1}{3}\right)^{3/2}.$$