Problem 26

(a) Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \tag{i}$$

where c is an arbitrary constant.

(b) Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \tag{ii}$$

corresponding to $g(t) = 0$.

(c) Show that y_2 is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher order linear equations have a pattern similar to Eq. (i).

Solution

Part (a)

The general linear equation in equation (1) is

$$y' + p(t)y = g(t). \tag{1}$$

It can be solved by multiplying both sides by an integrating factor μ.

$$\mu = \exp\left(\int^t p(s)\,ds\right)$$

Proceed with the multiplication.

$$\exp\left(\int^t p(s)\,ds\right) y' + p(t)\exp\left(\int^t p(s)\,ds\right) y = g(t)\exp\left(\int^t p(s)\,ds\right)$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\int^t p(s)\,ds\right) y\right] = g(t)\exp\left(\int^t p(s)\,ds\right)$$

Integrate both sides with respect to t.

$$\exp\left(\int^t p(s)\,ds\right) y = \int^t g(r)\exp\left(\int^r p(s)\,ds\right) dr + c$$

Divide both sides by $e^{\int^t p(s)\,ds}$ to solve for y.

$$y(t) = \exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right) dr + c\exp\left(-\int^t p(s)\,ds\right)$$

Therefore, $y(t)$ can be written in the form $cy_1(t) + y_2(t)$, where

$$y_1(t) = \exp\left(-\int^t p(s)\,ds\right)$$
$$y_2(t) = \exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right) dr.$$

Part (b)

The aim is to show that $y_1(t)$ satisfies $y' + p(t)y = 0$.

$$\left[\exp\left(-\int^t p(s)\,ds\right)\right]' + p(t)\left[\exp\left(-\int^t p(s)\,ds\right)\right] \stackrel{?}{=} 0$$

Apply the chain rule to take the derivative of the exponential function.

$$\exp\left(-\int^t p(s)\,ds\right)\frac{d}{dt}\left[-\int^t p(s)\,ds\right] + p(t)\exp\left(-\int^t p(s)\,ds\right) \stackrel{?}{=} 0$$

Bring the minus sign in front.

$$-\exp\left(-\int^t p(s)\,ds\right)\frac{d}{dt}\int^t p(s)\,ds + p(t)\exp\left(-\int^t p(s)\,ds\right) \stackrel{?}{=} 0$$

Use the fundamental theorem of calculus to differentiate the integral.

$$\begin{aligned} -\exp\left(-\int^t p(s)\,ds\right)p(t) + p(t)\exp\left(-\int^t p(s)\,ds\right) &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

Part (c)

The aim is to show that $y_2(t)$ satisfies $y' + p(t)y = g(t)$.

$$\begin{aligned} &\left[\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr\right]' \\ &\qquad + p(t)\left[\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr\right] \stackrel{?}{=} g(t) \end{aligned}$$

Use the product rule to evaluate the derivative.

$$\begin{aligned} &\left[\exp\left(-\int^t p(s)\,ds\right)\right]'\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \\ &\qquad + \exp\left(-\int^t p(s)\,ds\right)\left[\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr\right]' \\ &\qquad\qquad + p(t)\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \stackrel{?}{=} g(t) \end{aligned}$$

Use the chain rule to differentiate the exponential function and the fundamental theorem of calculus to differentiate the integral.

$$\begin{aligned} &\exp\left(-\int^t p(s)\,ds\right)\frac{d}{dt}\left[-\int^t p(s)\,ds\right]\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \\ &\qquad + \exp\left(-\int^t p(s)\,ds\right)\left[g(t)\exp\left(\int^t p(s)\,ds\right)\right] \\ &\qquad\qquad + p(t)\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \stackrel{?}{=} g(t) \end{aligned}$$

Bring the minus sign in front of the first term and combine the exponential functions in the second term.

$$\begin{aligned} -\exp\left(-\int^t p(s)\,ds\right)\left[\frac{d}{dt}\int^t p(s)\,ds\right]\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \\ + g(t) \\ + p(t)\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \stackrel{?}{=} g(t) \end{aligned}$$

Use the fundamental theorem of calculus once more in the first term.

$$\begin{aligned} -\exp\left(-\int^t p(s)\,ds\right)[p(t)]\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \\ + g(t) \\ + p(t)\exp\left(-\int^t p(s)\,ds\right)\int^t g(r)\exp\left(\int^r p(s)\,ds\right)dr \stackrel{?}{=} g(t) \end{aligned}$$

The first and third terms cancel, leaving only $g(t)$ on the left side.

$$g(t) = g(t)$$