Problem 30

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

$$y' = \epsilon y - \sigma y^3, \qquad \epsilon > 0 \text{ and } \sigma > 0.$$

This equation occurs in the study of the stability of fluid flow.

Solution

Bring ϵy to the left side.

$$y' - \epsilon y = -\sigma y^3$$

Divide both sides by y^3.

$$y^{-3}y' - \epsilon y^{-2} = -\sigma \tag{1}$$

Make the substitution $u = y^{-2}$. Now differentiate both sides of it with respect to t to find y' in terms of this new variable.

$$\frac{du}{dt} = (-2)y^{-3} \cdot \frac{dy}{dt}$$

Divide both sides by -2.

$$-\frac{1}{2}\frac{du}{dt} = y^{-3}y'$$

Substitute this result and $u = y^{-2}$ into equation (1).

$$-\frac{1}{2}\frac{du}{dt} - \epsilon u = -\sigma$$

Multiply both sides by -2.

$$\frac{du}{dt} + 2\epsilon u = 2\sigma$$

This ODE can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t 2\epsilon\, ds\right) = e^{2\epsilon t}$$

Proceed with the multiplication.

$$e^{2\epsilon t}\frac{du}{dt} + 2\epsilon e^{2\epsilon t}u = 2\sigma e^{2\epsilon t}$$

The left side can be written as $d/dt(Iu)$ by the product rule.

$$\frac{d}{dt}(e^{2\epsilon t}u) = 2\sigma e^{2\epsilon t}$$

Integrate both sides with respect to t.

$$e^{2\epsilon t}u = \frac{\sigma}{\epsilon}e^{2\epsilon t} + C$$

Divide both sides by $e^{2\epsilon t}$.

$$u(t) = \frac{\sigma}{\epsilon} + Ce^{-2\epsilon t}$$

Now that u is solved for, replace it with y^{-2} and solve for y.

$$y^{-2} = \frac{\sigma}{\epsilon} + Ce^{-2\epsilon t}$$

$$y^2 = \frac{1}{\frac{\sigma}{\epsilon} + Ce^{-2\epsilon t}}$$

Therefore,

$$y(t) = \pm\sqrt{\frac{\epsilon}{\sigma + C\epsilon e^{-2\epsilon t}}}.$$