Problem 14

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' = 2ty^2, \qquad y(0) = y_0$$

Solution

Method Using the Chain Rule

$$y' = 2ty^2$$

Divide both sides by y^2.

$$\frac{y'}{y^2} = 2t$$

The left side can be written as $d/dt(-1/y)$ by the chain rule.

$$\frac{d}{dt}\left(-\frac{1}{y}\right) = 2t$$

Integrate both sides with respect to t.

$$-\frac{1}{y} = t^2 + C_1$$

Apply the initial condition now to determine C_1.

$$-\frac{1}{y_0} = C_1$$

So then the previous equation becomes

$$-\frac{1}{y} = t^2 - \frac{1}{y_0}.$$

Therefore,

$$\begin{aligned} y(t) &= -\frac{1}{t^2 - \frac{1}{y_0}} \\ &= \frac{1}{\frac{1}{y_0} - t^2} \\ &= \frac{y_0}{1 - y_0 t^2}. \end{aligned}$$

Note that the solution blows up in a finite amount of time if y_0 is positive, specifically

$$1 - y_0 t^2 = 0 \quad \to \quad t = \pm\sqrt{\frac{1}{y_0}}.$$

No such thing occurs, though, if y_0 is not positive.

Method By Separating Variables

$$\frac{dy}{dt} = 2ty^2$$

Solve the ODE by separating variables.

$$\frac{dy}{y^2} = 2t\,dt$$

Integrate both sides.

$$-\frac{1}{y} = t^2 + C_2$$

Apply the initial condition now to determine C_2.

$$-\frac{1}{y_0} = C_2$$

So then the previous equation becomes

$$-\frac{1}{y} = t^2 - \frac{1}{y_0}.$$

Therefore,

$$\begin{aligned} y(t) &= -\frac{1}{t^2 - \frac{1}{y_0}} \\ &= \frac{1}{\frac{1}{y_0} - t^2} \\ &= \frac{y_0}{1 - y_0 t^2}. \end{aligned}$$

According to Theorem 2.4.2, a unique solution to

$$y' = f(t, y), \qquad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that f and $\partial f/\partial y$ are continuous in a rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ that contains (t_0, y_0). In this exercise

$$f(t, y) = 2ty^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4ty.$$

Both f and $\partial f/\partial y$ are continuous everywhere except where the solution for y blows up. If $y_0 > 0$, then a unique solution exists in an interval within

$$-\sqrt{\frac{1}{y_0}} < t < \sqrt{\frac{1}{y_0}},$$

and if $y_0 \leq 0$, then a unique solution exists in an interval within

$$-\infty < t < \infty.$$