Problem 15

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' + y^3 = 0, \qquad y(0) = y_0$$

Solution

Method Using the Chain Rule

Bring y^3 to the right side.

$$y' = -y^3$$

Divide both sides by y^3.

$$\frac{y'}{y^3} = -1$$

The left side can be written as $d/dt[-1/(2y^2)]$ by the chain rule.

$$\frac{d}{dt}\left(-\frac{1}{2y^2}\right) = -1$$

Integrate both sides with respect to t.

$$-\frac{1}{2y^2} = -t + C_1$$

Apply the initial condition now to determine C_1.

$$-\frac{1}{2y_0^2} = C_1$$

So then the previous equation becomes

$$-\frac{1}{2y^2} = -t - \frac{1}{2y_0^2}$$

$$\frac{1}{y^2} = 2t + \frac{1}{y_0^2}$$

$$y^2 = \frac{1}{2t + \frac{1}{y_0^2}}$$

$$= \frac{y_0^2}{2ty_0^2 + 1}.$$

Therefore, taking the square root of both sides,

$$y(t) = \frac{y_0}{\sqrt{2ty_0^2 + 1}}.$$

The positive root was chosen so that the initial condition remains satisfied. Note that the solution blows up in a finite amount of time if

$$2ty_0^2 + 1 = 0 \quad \rightarrow \quad t = -\frac{1}{2y_0^2}.$$

Method By Separating Variables

Bring y^3 to the right side.

$$\frac{dy}{dt} = -y^3$$

Separate variables.

$$\frac{dy}{y^3} = -dt$$

Integrate both sides.

$$-\frac{1}{2y^2} = -t + C_2$$

Apply the initial condition now to determine C_2.

$$-\frac{1}{2y_0^2} = C_2$$

So then the previous equation becomes

$$-\frac{1}{2y^2} = -t - \frac{1}{2y_0^2}$$

$$\frac{1}{y^2} = 2t + \frac{1}{y_0^2}$$

$$\begin{aligned} y^2 &= \frac{1}{2t + \frac{1}{y_0^2}} \\ &= \frac{y_0^2}{2ty_0^2 + 1}. \end{aligned}$$

Therefore, taking the square root of both sides,

$$y(t) = \frac{y_0}{\sqrt{2ty_0^2 + 1}}.$$

The positive root was chosen so that the initial condition remains satisfied. According to Theorem 2.4.2, a unique solution to

$$y' = f(t, y), \qquad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that f and $\partial f/\partial y$ are continuous in a rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ that contains (t_0, y_0). In this exercise

$$f(t, y) = -y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = -3y^2.$$

Both f and $\partial f/\partial y$ are continuous everywhere except where the solution for y blows up. Since the initial condition is given at $t = 0$, a unique solution exists in an interval within

$$-\frac{1}{2y_0^2} < t < \infty.$$