Problem 22

(a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \qquad y(2) = -1.$$

Where are these solutions valid?

(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.

(c) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

Solution

Part (a)

Plug both solutions into the ODE and see if both sides are equal.

$$\begin{aligned}
(1-t)' &\stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4(1-t)}}{2} \\
-1 &\stackrel{?}{=} \frac{-t + \sqrt{t^2 - 4t + 4}}{2} \\
-1 &\stackrel{?}{=} \frac{-t + \sqrt{(t-2)^2}}{2} \\
-1 &\stackrel{?}{=} \frac{-t + t - 2}{2} \\
-1 &= -1
\end{aligned}$$

$$\begin{aligned}
\left(-\frac{t^2}{4}\right)' &\stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4\left(-\frac{t^2}{4}\right)}}{2} \\
-\frac{t}{2} &\stackrel{?}{=} \frac{-t + \sqrt{t^2 - t^2}}{2} \\
-\frac{t}{2} &= -\frac{t}{2}
\end{aligned}$$

$$y_1(2) = 1 - 2 = -1 \qquad\qquad y_2(2) = -\frac{2^2}{4} = -1$$

Note that $\sqrt{(t-2)^2} = |t-2| = t-2$ only if $t - 2 \geq 0$. As a result, the solution $y_1(t)$ is only valid for $t \geq 2$. There are no restrictions for the solution $y_2(t)$, so it is valid for $-\infty < t < \infty$.

Part (b)

According to Theorem 2.4.2, a unique solution to

$$y' = f(t, y), \qquad y(t_0) = y_0$$

exists in some interval $t_0 - h < t < t_0 + h$ within $\alpha < t < \beta$, provided that f and $\partial f/\partial y$ are continuous in a rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ that contains (t_0, y_0). In this exercise

$$f(t, y) = \frac{-t + \sqrt{t^2 + 4y}}{2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2 + 4y}}.$$

$\partial f/\partial y$ is discontinuous at $t = 2$ and $y = -1$, so there is no guarantee of a unique solution. That's why we have $y_1(t)$ and $y_2(t)$.

Part (c)

Show that $y = ct + c^2$ satisfies the ODE.

$$\begin{aligned} y' &\stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4y}}{2} \\ (ct + c^2)' &\stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4(ct + c^2)}}{2} \\ c &\stackrel{?}{=} \frac{-t + \sqrt{t^2 + 4ct + 4c^2}}{2} \\ c &\stackrel{?}{=} \frac{-t + \sqrt{(t + 2c)^2}}{2} \\ c &\stackrel{?}{=} \frac{-t + t + 2c}{2} \\ c &= c \end{aligned}$$

Note that $\sqrt{(t + 2c)^2} = |t + 2c| = t + 2c$ only if $t + 2c \geq 0$. As a result, the solution $y(t) = ct + c^2$ is only valid for $t \geq -2c$. No value of c gives $y_2(t)$ because there's no t^2 term.