Problem 32

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0.

Solve the initial value problem

$$y' + 2y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

Solution

Split up the problem over two intervals of time. Solve each ODE by multiplying both sides by the integrating factor $I = e^{2t}$.

$\boldsymbol{0 \le t \le 1}$

$$\begin{aligned} y' + 2y &= 1 \\ e^{2t}y' + 2e^{2t}y &= e^{2t} \\ \frac{d}{dt}(e^{2t}y) &= e^{2t} \\ e^{2t}y &= \frac{1}{2}e^{2t} + C_1 \\ y(t) &= \frac{1}{2} + C_1e^{-2t} \end{aligned}$$

$\boldsymbol{t > 1}$

$$\begin{aligned} y' + 2y &= 0 \\ e^{2t}y' + 2e^{2t}y &= 0 \\ \frac{d}{dt}(e^{2t}y) &= 0 \\ e^{2t}y &= C_2 \\ y(t) &= C_2e^{-2t} \end{aligned}$$

Since $t = 0$ is part of the $0 \le t \le 1$ interval, use the initial condition $y(0) = 0$ to determine C_1.

$$y(0) = \frac{1}{2} + C_1 = 0 \quad \rightarrow \quad C_1 = -\frac{1}{2}$$

Then use the fact that y is continuous at $t = 1$ to determine C_2.

$$\begin{aligned} \frac{1}{2} + C_1e^{-2(1)} &= C_2e^{-2(1)} \\ \frac{1}{2} - \frac{1}{2}e^{-2} &= C_2e^{-2} \quad \rightarrow \quad C_2 = \frac{1}{2}e^2 - \frac{1}{2} \end{aligned}$$

Therefore,

$$y(t) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2}e^{-2t} & \text{if } 0 \le t \le 1 \\ \left(\dfrac{1}{2}e^2 - \dfrac{1}{2}\right)e^{-2t} & \text{if } t > 1 \end{cases}.$$