Problem 33

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0.

Solve the initial value problem

$$y' + p(t)y = 0, \qquad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

Solution

Split up the problem over two intervals of time. Solve each ODE by multiplying both sides by the integrating factors, $I_1 = e^{2t}$ and $I_2 = e^t$, respectively.

$\mathbf{0 \le t \le 1}$	$\mathbf{t > 1}$
$y' + 2y = 0$	$y' + y = 0$
$e^{2t}y' + 2e^{2t}y = 0$	$e^t y' + e^t y = 0$
$\frac{d}{dt}(e^{2t}y) = 0$	$\frac{d}{dt}(e^t y) = 0$
$e^{2t}y = C_1$	$e^t y = C_2$
$y(t) = C_1 e^{-2t}$	$y(t) = C_2 e^{-t}$

Since $t = 0$ is part of the $0 \le t \le 1$ interval, use the initial condition $y(0) = 1$ to determine C_1.

$$y(0) = C_1 = 1$$

Then use the fact that y is continuous at $t = 1$ to determine C_2.

$$\begin{aligned} C_1 e^{-2(1)} &= C_2 e^{-(1)} \\ e^{-2} &= C_2 e^{-1} \quad \rightarrow \quad C_2 = e^{-1} \end{aligned}$$

Therefore,

$$y(t) = \begin{cases} e^{-2t} & \text{if } 0 \le t \le 1 \\ e^{-1}e^{-t} & \text{if } t > 1 \end{cases}.$$