Problem 7

Semistable Equilibrium Solutions. Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side depart from it (see Figure 2.5.9). In this case the equilibrium solution is said to be **semistable**.

(a) Consider the equation

$$dy/dt = k(1-y)^2, \tag{i}$$

where k is a positive constant. Show that $y = 1$ is the only critical point, with the corresponding equilibrium solution $\phi(t) = 1$.

(b) Sketch $f(y)$ versus y. Show that y is increasing as a function of t for $y < 1$ and also for $y > 1$. The phase line has upward-pointing arrows both below and above $y = 1$. Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore, $\phi(t) = 1$ is semistable.

(c) Solve Eq. (i) subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part (b).

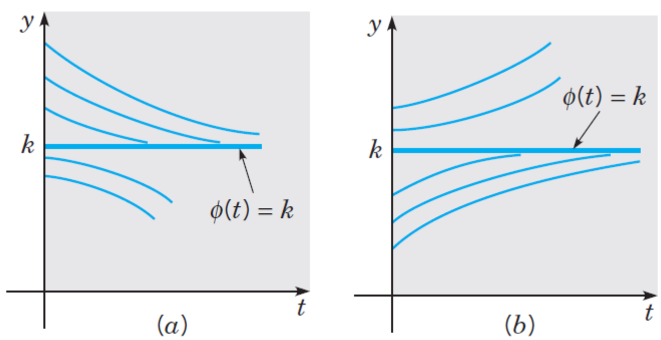


FIGURE 2.5.9 In both cases the equilibrium solution $\phi(t) = k$ is semistable. (a) $dy/dt \le 0$; (b) $dy/dt \ge 0$.

Solution

Part (a)

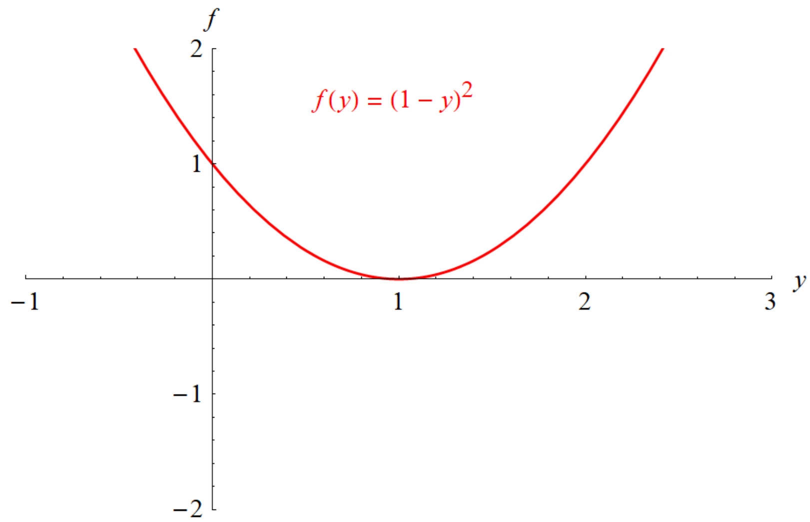
The critical points (equilibrium solutions) are obtained by solving $f(y) = k(1-y)^2 = 0$ for y.

$$k(1-y)^2 = 0$$

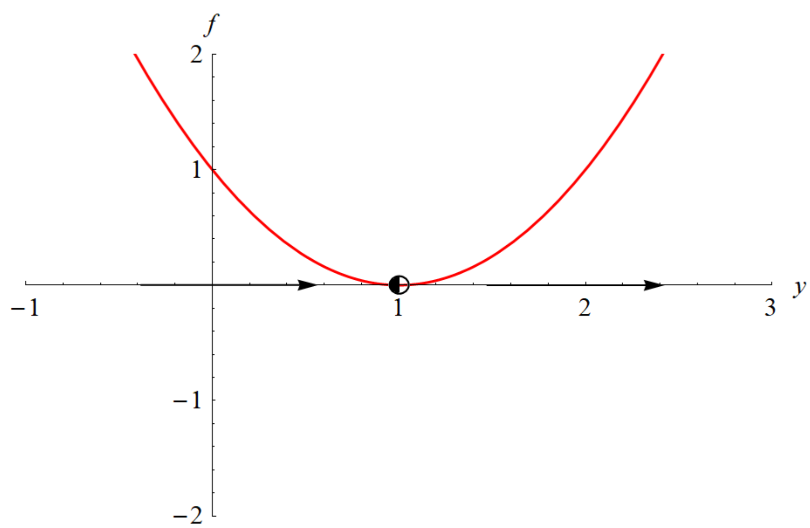
$$y = \{1\}$$

Part (b)

Below is the graph of $f(y)$ versus y for $k = 1$.

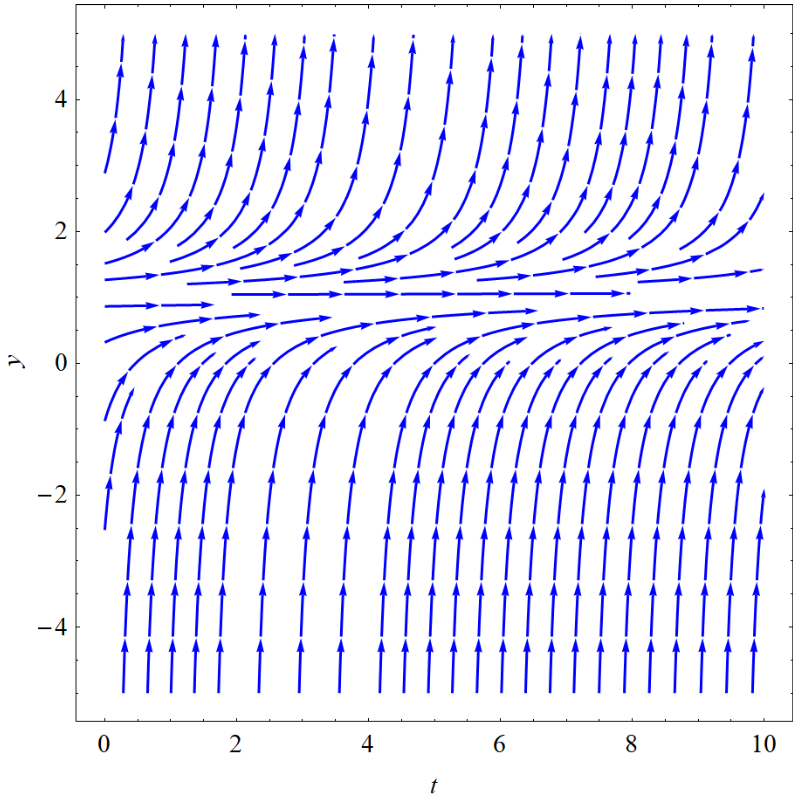


As indicated by the half-filled circle below, $y = 1$ is stable coming from the left but unstable from the right. It is said to be semistable.



The arrows pointing right on the y-axis (phase line) mean that y is increasing in time.

Some possible solution curves in the ty-plane for $t \geq 0$ are shown below. At every point, they are tangent to the direction field vectors $\langle 1, (1-y)^2 \rangle$.



Part (c)

$$\frac{dy}{dt} = k(1-y)^2$$

Solve the ODE by separating variables.

$$\frac{dy}{(1-y)^2} = k\,dt$$

Integrate both sides.

$$\int^y \frac{ds}{(1-s)^2} = kt + C$$

Make the substitution $u = 1 - s$ so that $du = -ds$ on the left.

$$\int^{1-y} \frac{-du}{u^2} = kt + C$$

$$\left.\frac{1}{u}\right|^{1-y} = kt + C$$

$$\frac{1}{1-y} = kt + C \tag{1}$$

Use the initial condition $y(0) = y_0$ to determine C.

$$\frac{1}{1-y_0} = C$$

So equation (1) becomes

$$\frac{1}{1-y} = kt + \frac{1}{1-y_0}.$$

Now solve for y.

$$\begin{aligned} 1 - y &= \frac{1}{kt + \frac{1}{1-y_0}} \\ &= \frac{1-y_0}{kt(1-y_0)+1} \end{aligned}$$

$$\begin{aligned} y(t) &= 1 - \frac{1-y_0}{kt(1-y_0)+1} \\ &= \frac{kt(1-y_0)+1-1+y_0}{kt(1-y_0)+1} \end{aligned}$$

Therefore,

$$y(t) = \frac{kt(1-y_0)+y_0}{kt(1-y_0)+1}.$$

Differentiate this solution with respect to t.

$$\begin{aligned} \frac{dy}{dt} &= \frac{k(1-y_0)[kt(1-y_0)+1] - k(1-y_0)[kt(1-y_0)+y_0]}{[kt(1-y_0)+1]^2} \\ &= \frac{k - 2ky_0 + ky_0^2}{[kt(1-y_0)+1]^2} \\ &= \frac{k(1-y_0)^2}{[kt(1-y_0)+1]^2} \end{aligned}$$

Because k is positive and the other terms are squared, dy/dt is always positive, which means y is always increasing.