Problem 14

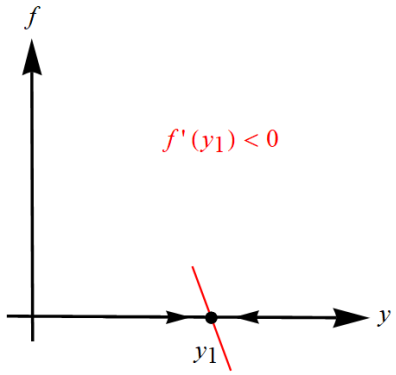
Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point—that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.

Solution

The ODE under consideration is

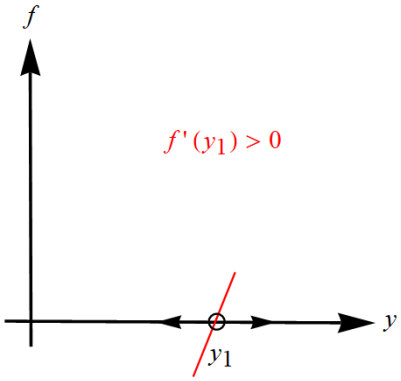
$$\frac{dy}{dt} = f(y).$$

Suppose that y_1 is a critical point, that is, $f(y_1) = 0$. If $f'(y_1) < 0$, then the slope of the curve going through $y = y_1$ is negative as illustrated below.



For values of y below y_1, f is positive, meaning that y is increasing in time. However, for values of y above y_1, f is negative, meaning that y is decreasing in time. Therefore, $y = y_1$ is a stable equilibrium point in this case.

On the other hand, if $f'(y_1) > 0$, then the slope of the curve going through $y = y_1$ is positive as illustrated below.



For values of y below y_1, f is negative, meaning that y is decreasing in time. However, for values of y above y_1, f is positive, meaning that y is increasing in time. Therefore, $y = y_1$ is an unstable equilibrium point in this case.