# Problem 20

**Harvesting a Renewable Resource.** Suppose that the population $y$ of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

Although it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. Problems 20 and 21 explore some of the questions involved in formulating a rational strategy for managing the fishery.[15]

At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population $y$: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $Ey$, where $E$ is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$dy/dt = r(1 - y/K)y - Ey. \tag{i}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.

(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield $Y$ of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort $E$ and the asymptotically stable population $y_2$. Find $Y$ as a function of the effort $E$; the graph of this function is known as the yield–effort curve.

(d) Determine $E$ so as to maximize $Y$ and thereby find the **maximum sustainable yield** $Y_m$.

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**Solution**

The equilibrium points are found by setting $dy/dt = 0$ and solving the equation for $y$.

$$r\left(1 - \frac{y}{K}\right)y - Ey = 0$$

$$y\left[r\left(1 - \frac{y}{K}\right) - E\right] = 0$$

$$y = 0 \qquad \text{or} \qquad r\left(1 - \frac{y}{K}\right) - E = 0$$

$$1 - \frac{y}{K} = \frac{E}{r}$$

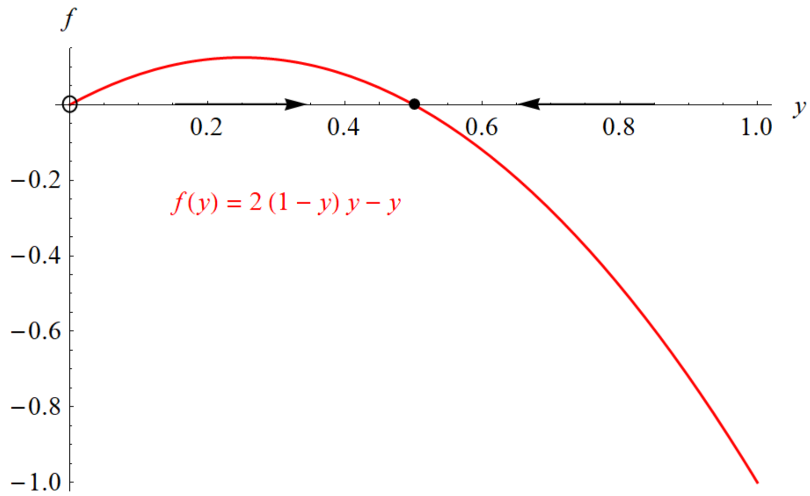
$$\frac{y}{K} = 1 - \frac{E}{r}$$

$$y = K\left(1 - \frac{E}{r}\right)$$

If $E < r$, then this second solution is positive and is a possible equilibrium population.

[15] An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

Plotting $f(y) = r(1 - y/K)y - Ey$ versus $y$ with $r = 2$, $K = 1$, and $E = 1$, we see that the equilibrium at $y = 0$ is unstable and the equilibrium at $y = K(1 - E/r)$ is stable.




The sustainable yield is the effort times the asymptotically stable population.

$$Y(E) = EK\left(1 - \frac{E}{r}\right)$$

To maximize the sustainable yield, take the derivative of it, set it equal to zero, and solve the resulting equation for $E$.

$$Y'(E) = K\left(1 - \frac{E}{r}\right) + EK\left(-\frac{1}{r}\right) = 0$$

$$K\left(1 - \frac{E}{r}\right) = \frac{EK}{r}$$

$$K(r - E) = EK$$

$$rK - EK = EK$$

$$2EK = rK$$

$$E = \frac{r}{2}$$

The maximum sustainable yield is therefore

$$\begin{aligned} Y_m &= Y\left(E = \frac{r}{2}\right) \\ &= \left(\frac{r}{2}\right)K\left(1 - \frac{1}{2}\right) \\ &= \frac{rK}{4}. \end{aligned}$$