Problem 24

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[16] Problems 22 through 24 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity. Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt, the rate at which the number of susceptibles declines, is given by

$$dx/dt = -[\beta + \mu(t)]x. \tag{i}$$

The first term on the right side of Eq. (i) is the rate at which susceptibles contract smallpox, and the second term is the rate at which they die from all other causes. Also

$$dn/dt = -\nu\beta x - \mu(t)n, \tag{ii}$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right side are the death rates due to smallpox and to all other causes, respectively.

(a) Let $z = x/n$, and show that z satisfies the initial value problem

$$dz/dt = -\beta z(1 - \nu z), \qquad z(0) = 1. \tag{iii}$$

Observe that the initial value problem (iii) does not depend on $\mu(t)$.

(b) Find $z(t)$ by solving Eq. (iii).

(c) Bernoulli estimated that $\nu = \beta = \frac{1}{8}$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: On the basis of the model just described and the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years, 7 months. He therefore supported the inoculation program.

Solution

[16] A standard source is the book by Bailey listed in the references. The models in Problems 22, 23, and 24 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

Part (a)

We have the two ODEs,

$$\begin{aligned}\frac{dx}{dt} &= -[\beta + \mu(t)]x \\ \frac{dn}{dt} &= -\nu\beta x - \mu(t)n.\end{aligned}$$

Make the substitution $z = x/n$ and differentiate both sides of it with respect to t.

$$\begin{aligned}\frac{dz}{dt} &= \frac{x'n - n'x}{n^2} \\ &= \frac{-[\beta + \mu(t)]nx - [-\nu\beta x - \mu(t)n]x}{n^2} \\ &= \frac{-\beta nx - \mu(t)nx + \nu\beta x^2 + \mu(t)nx}{n^2} \\ &= \frac{-\beta nx + \nu\beta x^2}{n^2} \\ &= -\beta\frac{x}{n} + \nu\beta\frac{x^2}{n^2} \\ &= -\beta z + \nu\beta z^2 \\ &= -\beta z(1 - \nu z)\end{aligned}$$

At $t = 0$ the members of the cohort are born, so $n(0)$ is this number of members. Also, none of these newborns have smallpox, so $x(0)$ is this same number. The initial condition associated with the ODE for z is then

$$z(0) = \frac{x(0)}{n(0)} = 1.$$

Part (b)

Solve the ODE for z by separating variables.

$$\frac{dz}{z(1 - \nu z)} = -\beta\,dt$$

Integrate both sides.

$$\int \frac{dz}{z(1 - \nu z)} = -\beta t + C$$

Use partial fraction decomposition on the left.

$$\int \left(\frac{1}{z} + \frac{\nu}{1 - \nu z}\right) dz = -\beta t + C$$

$$\int \left(\frac{1}{z} - \frac{1}{z - \frac{1}{\nu}}\right) dz = -\beta t + C$$

$$\ln|z| - \ln\left|z - \frac{1}{\nu}\right| = -\beta t + C$$

Combine the logarithms.

$$\ln\left|\frac{z}{z-\frac{1}{\nu}}\right| = -\beta t + C$$

Apply the initial condition $z(0) = 1$ now to determine C.

$$\ln\left|\frac{1}{1-\frac{1}{\nu}}\right| = C$$

As a result, the previous equation becomes

$$\ln\left|\frac{z}{z-\frac{1}{\nu}}\right| = -\beta t + \ln\left|\frac{1}{1-\frac{1}{\nu}}\right|.$$

Exponentiate both sides.

$$\begin{aligned}\left|\frac{z}{z-\frac{1}{\nu}}\right| &= \exp\left(-\beta t + \ln\left|\frac{1}{1-\frac{1}{\nu}}\right|\right) \\ &= \exp(-\beta t)\exp\left(\ln\left|\frac{1}{1-\frac{1}{\nu}}\right|\right) \\ &= e^{-\beta t}\left|\frac{1}{1-\frac{1}{\nu}}\right|\end{aligned}$$

Since ν is positive, the absolute value sign on the right can be dropped. For the initial condition to remain satisfied, the absolute value sign on the left needs to be removed as well.

$$\frac{z}{z-\frac{1}{\nu}} = \frac{e^{-\beta t}}{1-\frac{1}{\nu}}$$

Solve for z.

$$z = z\cdot\frac{e^{-\beta t}}{1-\frac{1}{\nu}} - \frac{1}{\nu}\cdot\frac{e^{-\beta t}}{1-\frac{1}{\nu}}$$

$$z\left(1-\frac{e^{-\beta t}}{1-\frac{1}{\nu}}\right) = -\frac{e^{-\beta t}}{\nu-1}$$

$$z\left(1-\frac{\nu e^{-\beta t}}{\nu-1}\right) = \frac{e^{-\beta t}}{1-\nu}$$

$$z\left(1+\frac{\nu e^{-\beta t}}{1-\nu}\right) = \frac{e^{-\beta t}}{1-\nu}$$

$$z\left(\frac{1-\nu+\nu e^{-\beta t}}{1-\nu}\right) = \frac{e^{-\beta t}}{1-\nu}$$

$$z(t) = \frac{e^{-\beta t}}{1-\nu+\nu e^{-\beta t}}$$

Part (c)

Set $\nu = 1/8$ and $\beta = 1/8$ and $t = 20$.

$$z(20) = \frac{e^{-20/8}}{1 - \frac{1}{8} + \frac{1}{8}e^{-20/8}} \approx 0.0927 = 9.27\%$$

Roughly one in ten individuals have not had smallpox by age 20.