Problem 25

Bifurcation Points. For an equation of the form

$$dy/dt = f(a, y), \tag{i}$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a bifurcation point, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 25 through 27 describe three types of bifurcations that can occur in simple equations of the form (i).

Consider the equation

$$dy/dt = a - y^2. \tag{ii}$$

(a) Find all of the critical points for Eq. (ii). Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$.

(b) Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.

(c) In each case sketch several solutions of Eq. (ii) in the ty-plane.

(d) If we plot the location of the critical points as a function of a in the ay-plane, we obtain Figure 2.5.10. This is called the **bifurcation diagram** for Eq. (ii). The bifurcation at $a = 0$ is called a **saddle–node bifurcation**. This name is more natural in the context of second order systems, which are discussed in Chapter 9.

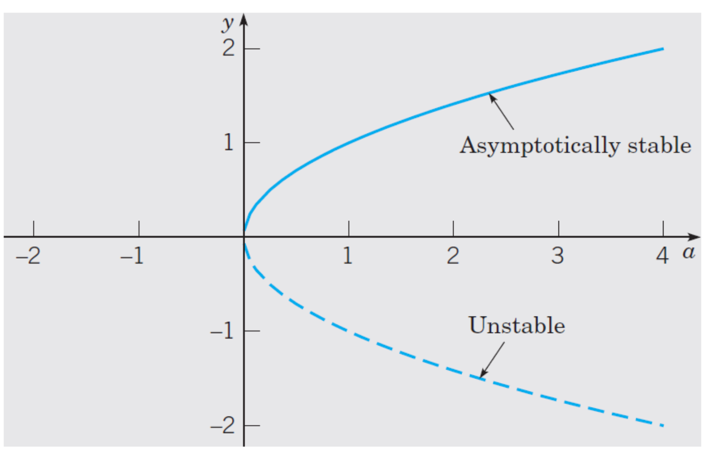


FIGURE 2.5.10 Bifurcation diagram for $y' = a - y^2$.

Solution

The critical points are found by setting $dy/dt = 0$ and solving the resulting equation for y.

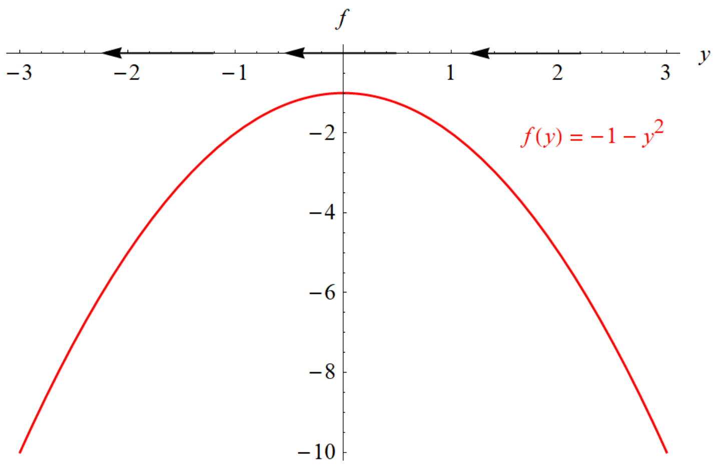
$$a - y^2 = 0$$

$$y^2 = a$$

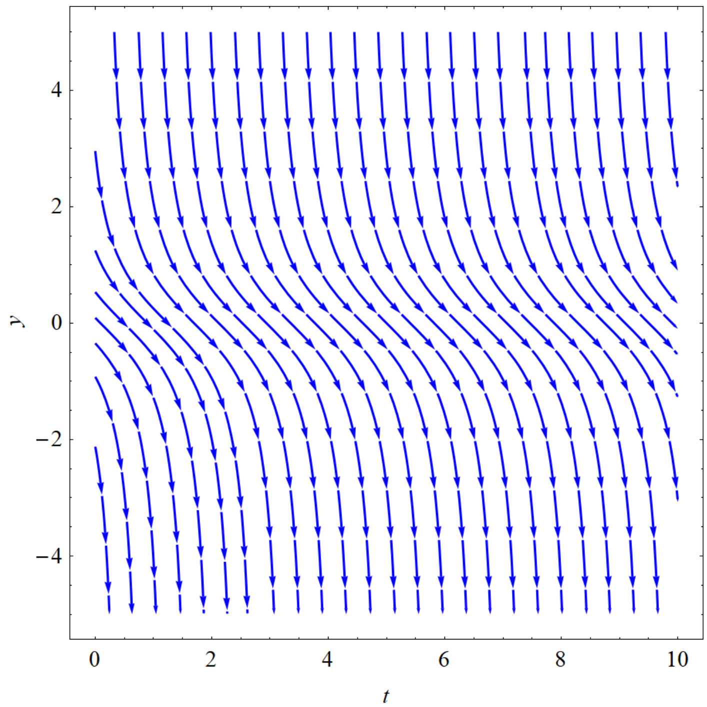
$$y = \{\pm\sqrt{a}\}$$

If a is negative, then there are no critical points; if a is zero, then there is one critical point; and if a is positive, then there are two critical points.

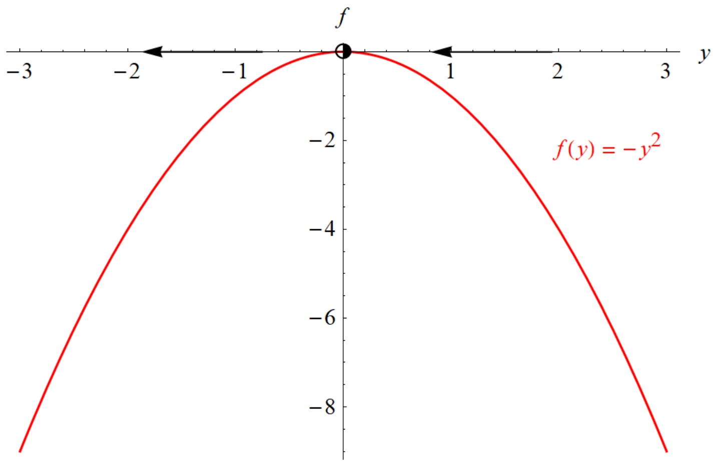
Suppose first that a is negative. The phase line is drawn below for $a = -1$.



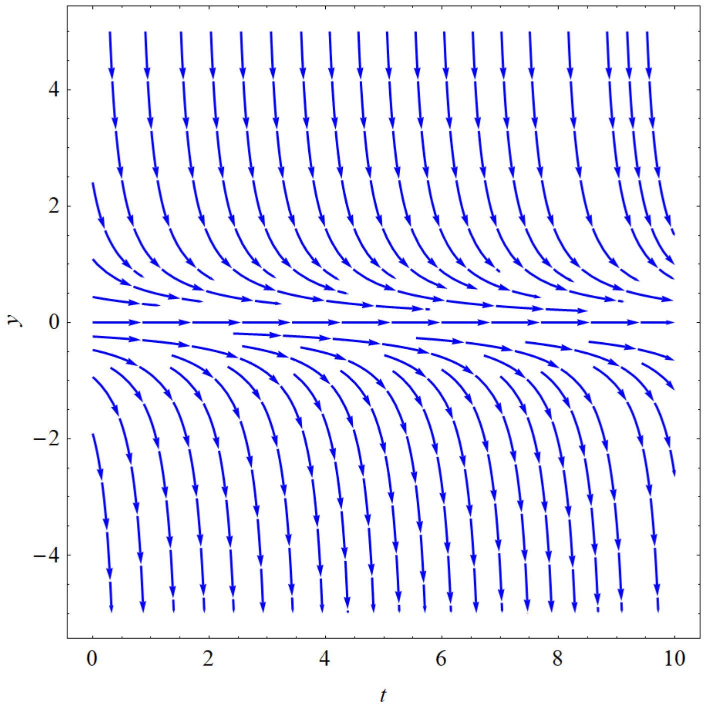
Solution curves in the ty-plane corresponding to $a = -1$ are tangent to the vectors in the direction field $\langle 1, -1 - y^2 \rangle$ at every point.



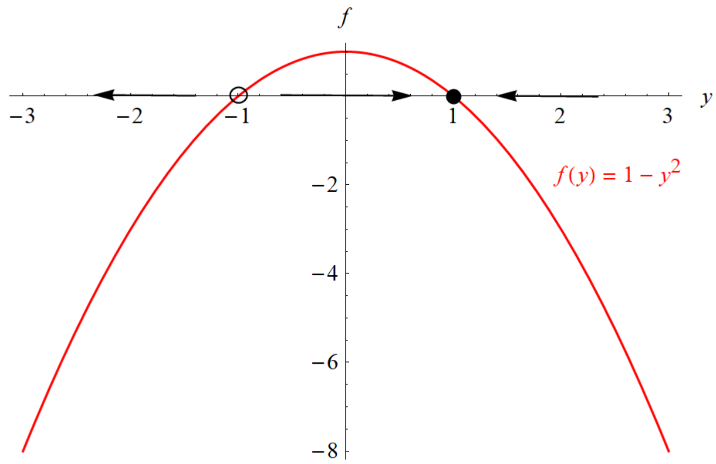
Suppose secondly that a is zero.



There is one critical point, and it is semistable. Solution curves in the ty-plane corresponding to $a = 0$ are tangent to the vectors in the direction field $\langle 1, -y^2 \rangle$ at every point.



Suppose thirdly that a is positive. The phase line is drawn below for $a = 1$.



Solution curves in the ty-plane corresponding to $a = 1$ are tangent to the vectors in the direction field $\langle 1, 1 - y^2 \rangle$ at every point.