Problem 26

Bifurcation Points. For an equation of the form

$$dy/dt = f(a, y), \tag{i}$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a bifurcation point, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 25 through 27 describe three types of bifurcations that can occur in simple equations of the form (i).

Consider the equation

$$dy/dt = ay - y^3 = y(a - y^2). \tag{iii}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case sketch several solutions of Eq. (iii) in the ty-plane.

(c) Draw the bifurcation diagram for Eq. (iii)—that is, plot the location of the critical points versus a. For Eq. (iii) the bifurcation point at $a = 0$ is called a **pitchfork bifurcation**. Your diagram may suggest why this name is appropriate.

Solution

The critical points are found by setting $dy/dt = 0$ and solving the resulting equation for y.

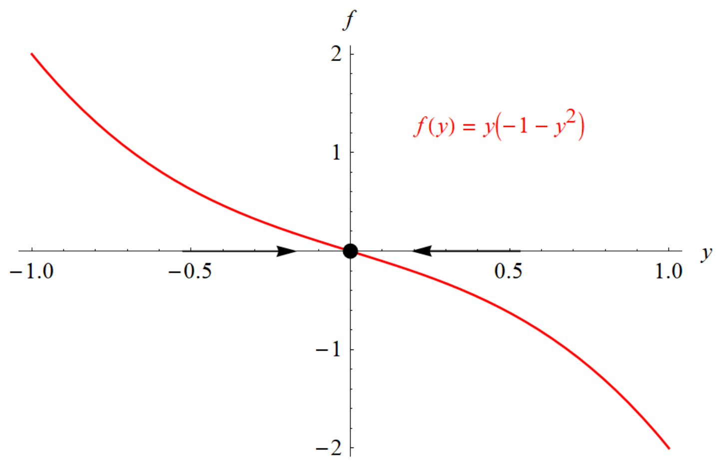
$$y(a - y^2) = 0$$

$$y = 0 \quad \text{or} \quad a - y^2 = 0$$

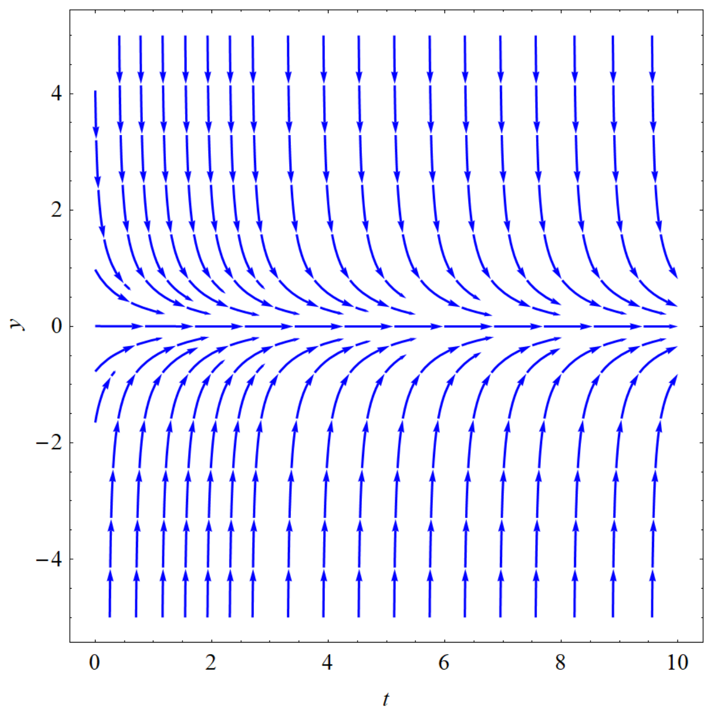
$$y = \{0, \pm\sqrt{a}\}$$

If a is negative, then there is only one critical point at $y = 0$; if a is zero, then there is still only one critical point at $y = 0$; and if a is positive, then there are three critical points.

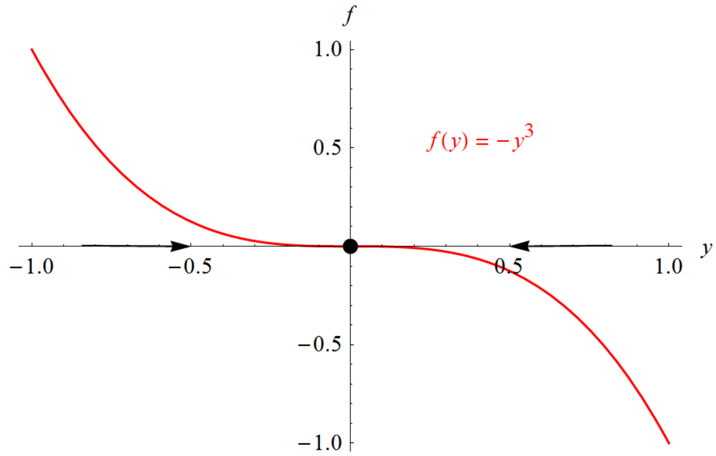
Suppose first that a is negative. The phase line is drawn below for $a = -1$.



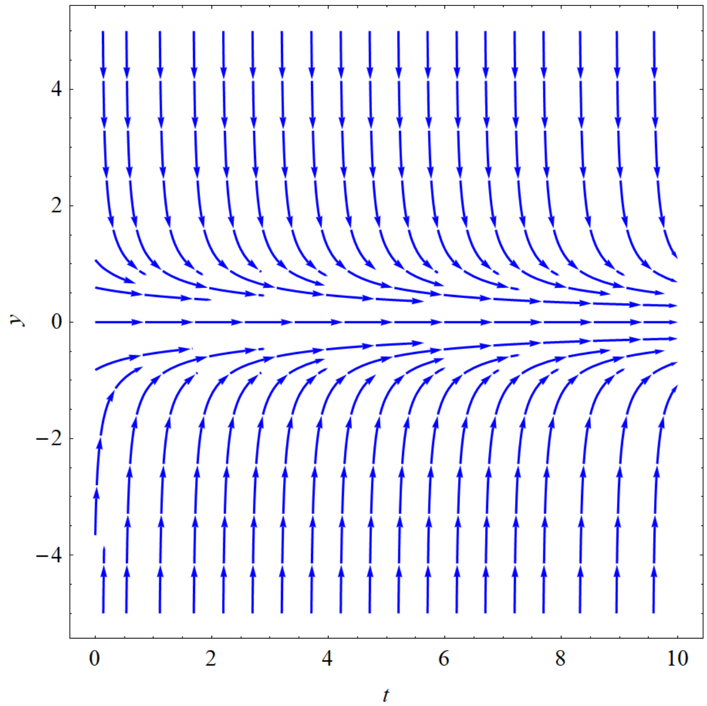
There is one critical point at $y = 0$, and it is stable. Solution curves in the ty-plane corresponding to $a = -1$ are tangent to the vectors in the direction field $\langle 1, y(-1 - y^2)\rangle$ at every point.



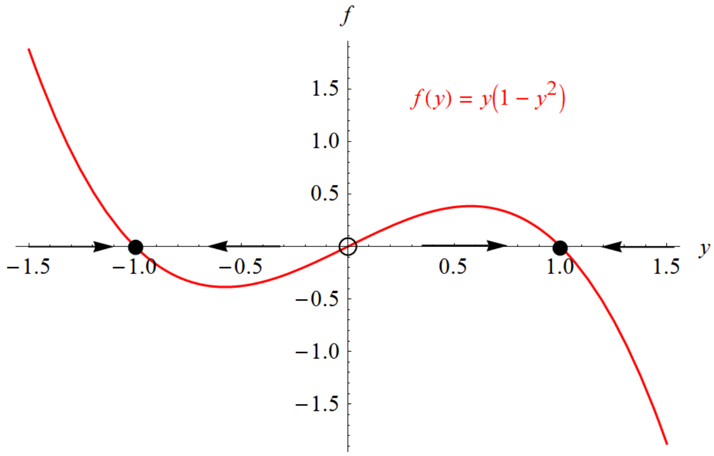
Suppose secondly that a is zero.



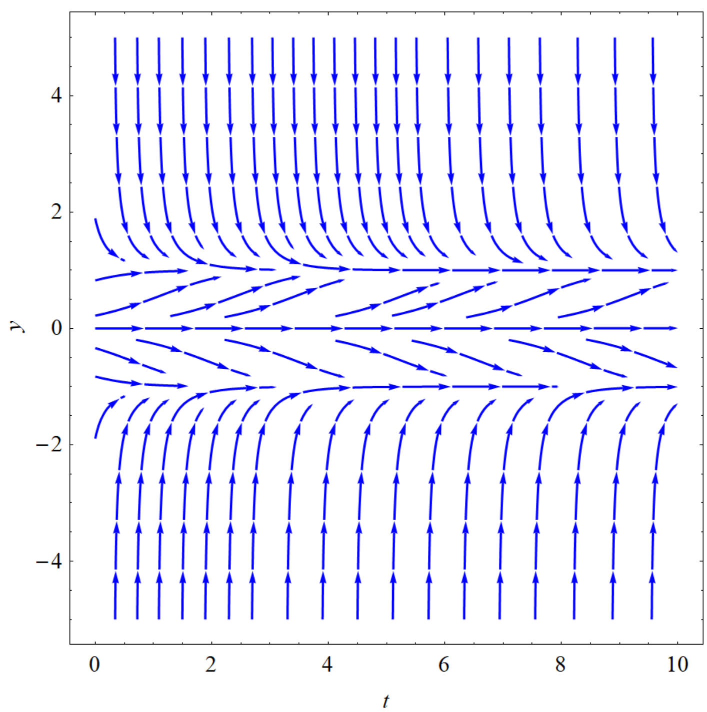
There is one critical point at $y = 0$, and it is stable. Solution curves in the ty-plane corresponding to $a = 0$ are tangent to the vectors in the direction field $\langle 1, -y^3 \rangle$ at every point.



Suppose thirdly that a is positive. The phase line is drawn below for $a = 1$.



There are now three critical points at $y = -\sqrt{a}$, $y = 0$, and $y = \sqrt{a}$, and they are stable, unstable, and stable, respectively. Solution curves in the ty-plane corresponding to $a = 1$ are tangent to the vectors in the direction field $\langle 1, y(1 - y^2)\rangle$ at every point.



This is the bifurcation diagram.