# Problem 27

**Bifurcation Points.** For an equation of the form

$$dy/dt = f(a, y), \tag{i}$$

where $a$ is a real parameter, the critical points (equilibrium solutions) usually depend on the value of $a$. As $a$ steadily increases or decreases, it often happens that at a certain value of $a$, called a bifurcation point, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 25 through 27 describe three types of bifurcations that can occur in simple equations of the form (i).

Consider the equation

$$dy/dt = ay - y^2 = y(a - y). \tag{iv}$$

(a) Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

(b) In each case sketch several solutions of Eq. (iv) in the $ty$-plane.

(c) Draw the bifurcation diagram for Eq. (iv). Observe that for Eq. (iv) there are the same number of critical points for $a < 0$ and $a > 0$ but that their stability has changed. For $a < 0$ the equilibrium solution $y = 0$ is asymptotically stable and $y = a$ is unstable, while for $a > 0$ the situation is reversed. Thus there has been an **exchange of stability** as $a$ passes through the bifurcation point $a = 0$. This type of bifurcation is called a **transcritical bifurcation**.

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### Solution

The critical points are found by setting $dy/dt = 0$ and solving the resulting equation for $y$.

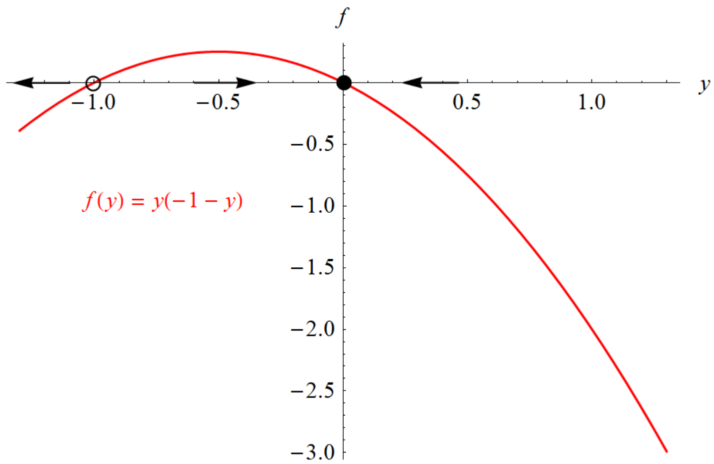
$$y(a - y) = 0$$

$$y = 0 \quad \text{or} \quad a - y = 0$$

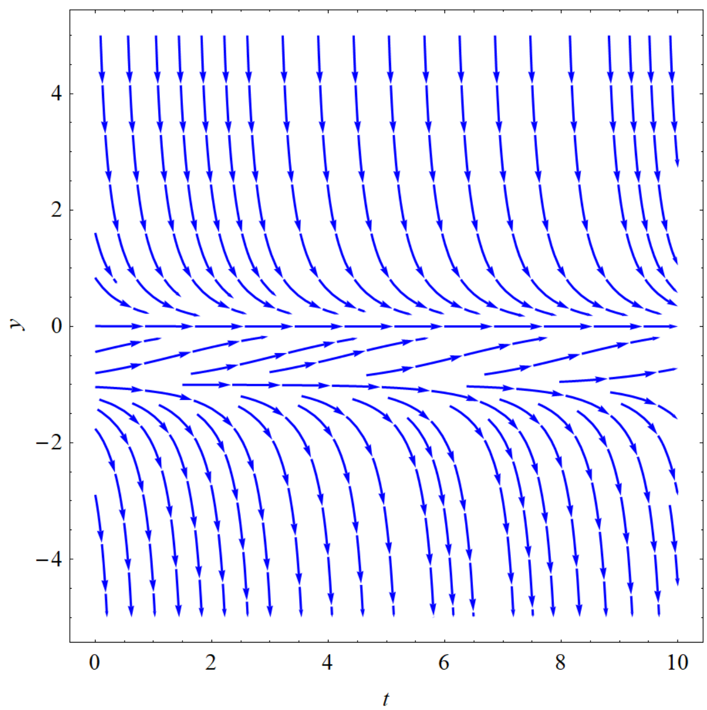
$$y = \{0, a\}$$

There are two critical points if $a$ is negative or positive, and there is a single critical point at $y = 0$ if $a$ is zero.

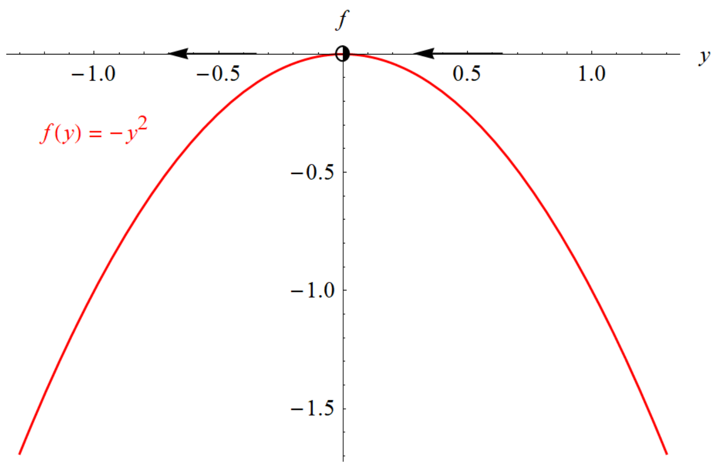
Suppose first that $a$ is negative. The phase line is drawn below for $a = -1$.



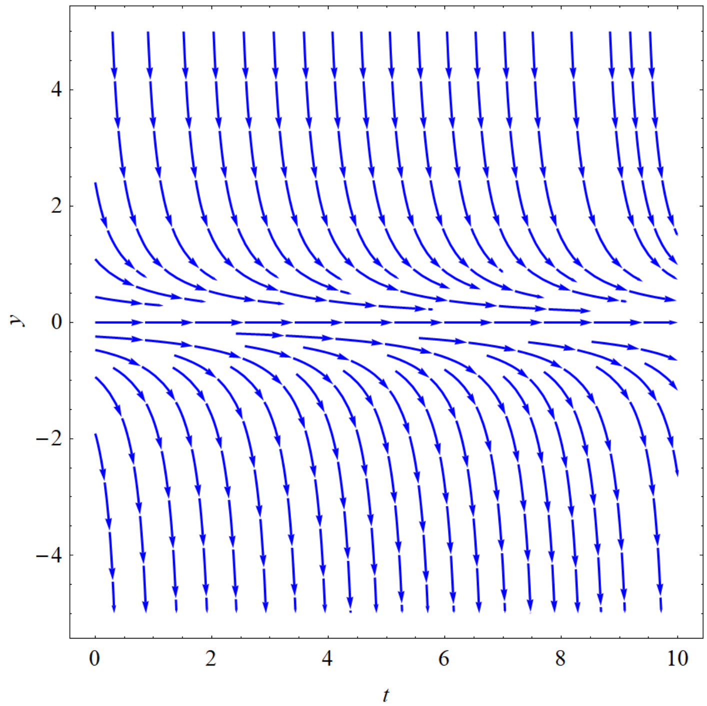

The critical point at $y = a$ is unstable, and the critical point at $y = 0$ is stable. Solution curves in the $ty$-plane corresponding to $a = -1$ are tangent to the vectors in the direction field $\langle 1, y(-1-y)\rangle$ at every point.



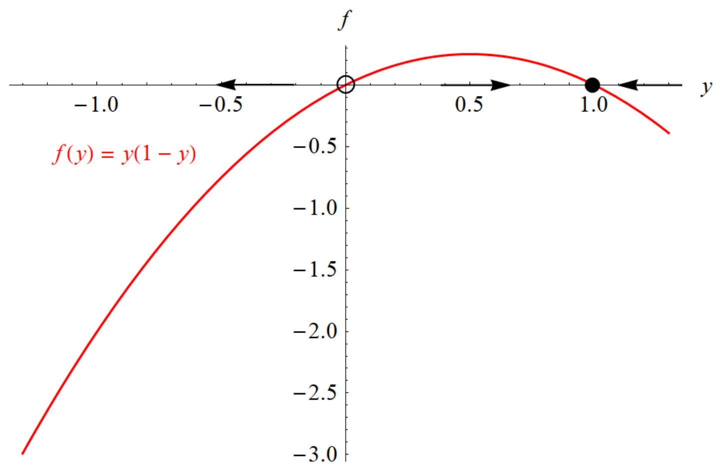
Suppose secondly that $a$ is zero.



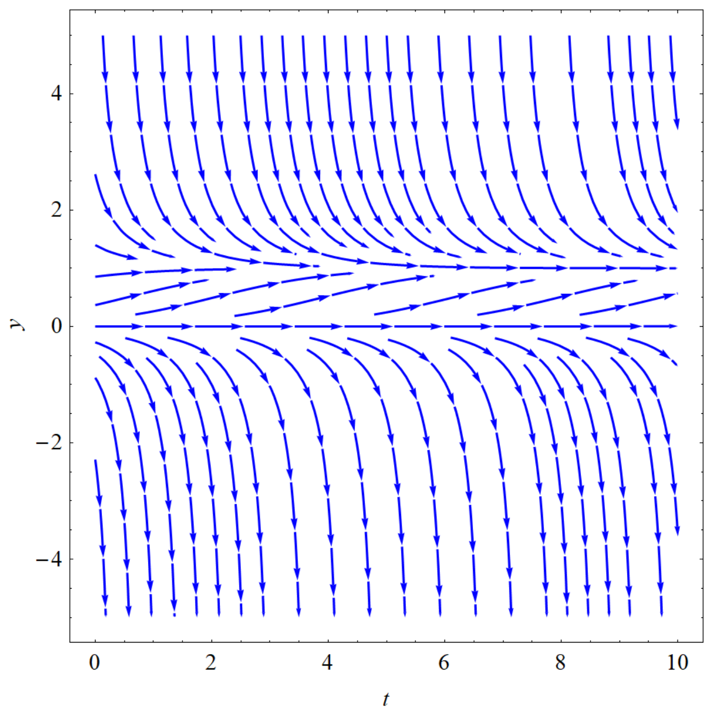

There is one critical point at $y = 0$, and it is semistable. Solution curves in the $ty$-plane corresponding to $a = 0$ are tangent to the vectors in the direction field $\langle 1, -y^2 \rangle$ at every point.



Suppose thirdly that $a$ is positive. The phase line is drawn below for $a = 1$.




The critical point at $y = a$ is now stable, and the critical point at $y = 0$ is now unstable. Solution curves in the $ty$-plane corresponding to $a = 1$ are tangent to the vectors in the direction field $\langle 1, y(1-y) \rangle$ at every point.



This is the bifurcation diagram.