Problem 16

Another equation that has been used to model population growth is the Gompertz[14] equation

$$dy/dt = ry\ln(K/y),$$

where r and K are positive constants.

(a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.

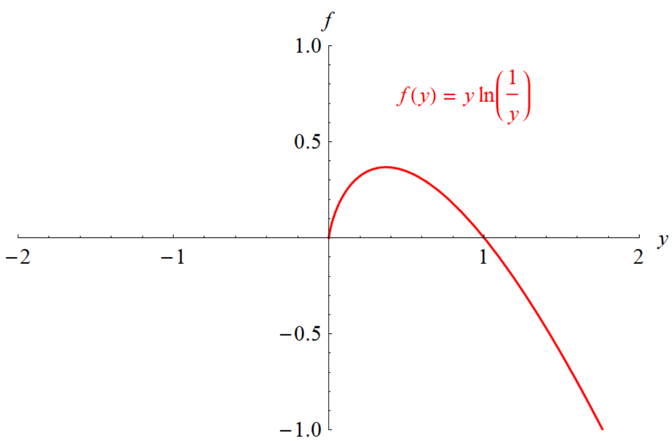
(b) For $0 \le y \le K$, determine where the graph of y versus t is concave up and where it is concave down.

(c) For each y in $0 < y \le K$, show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.

Solution

Part (a)

In this problem $f(y) = ry\ln(K/y)$. Below is a plot of $f(y)$ versus y for $r = 1$ and $K = 1$.



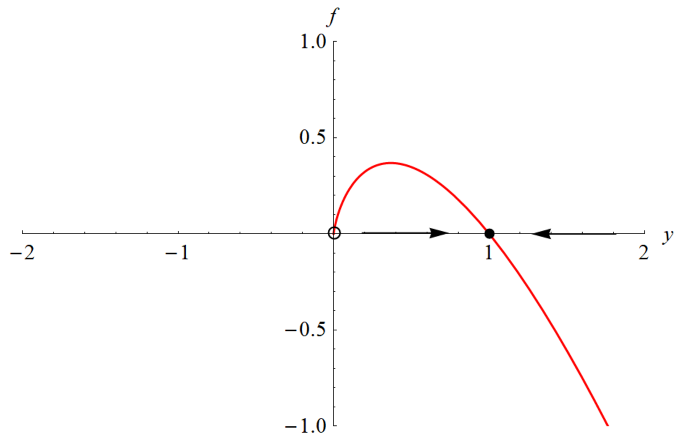
The critical points are obtained by solving $f(y) = 0$ for y.

$$ry\ln\frac{K}{y} = 0$$

$$y = \{0, K\}$$

[14]Benjamin Gompertz (1779–1865) was an English actuary. He developed his model for population growth, published in 1825, in the course of constructing mortality tables for his insurance company.

$y = 0$ is an unstable equilibrium solution, and $y = K$ is a stable equilibrium solution as indicated below by the open and closed circles, respectively.



Part (b)

To determine the concavity of the graph, we will calculate the second derivative of y with respect to t.

$$\frac{dy}{dt} = ry\ln\frac{K}{y}$$

Differentiate both sides with respect to t, using the chain rule on the right side.

$$\begin{aligned}\frac{d^2y}{dt^2} &= ry'\ln\frac{K}{y} + ry\frac{1}{\frac{K}{y}}\left(-\frac{K}{y^2}\right)y' \\ &= ry'\ln\frac{K}{y} - ry' \\ &= ry'\left(\ln\frac{K}{y} - 1\right) \\ &= r^2y\ln\frac{K}{y}\left(\ln\frac{K}{y} - 1\right)\end{aligned}$$

The graph is concave up wherever the second derivative is positive and concave down wherever the second derivative is negative. By inspection, we see that the quantity $r^2y\ln(K/y)$ is positive for $0 < y < K$. On the other hand, the quantity in parentheses is only positive when

$$\ln\frac{K}{y} - 1 > 0$$

$$\ln\frac{K}{y} > 1$$

$$\frac{K}{y} > e$$

$$y < \frac{K}{e}.$$

Therefore,

$$\text{Concave Up:} \qquad 0 < y < \frac{K}{e}$$
$$\text{Concave Down:} \qquad \frac{K}{e} < y < K.$$

The second derivative is zero at $y = 0$, $y = K/e$, and $y = K$, so these are inflection points.

Part (c)

Gompertz Equation	**Logistic Equation**
$\frac{dy}{dt} = ry \ln \frac{K}{y}$	$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$

Here we have to show that

$$ry \ln \frac{K}{y} \geq ry\left(1 - \frac{y}{K}\right).$$

Divide both sides by ry.

$$\ln \frac{K}{y} \stackrel{?}{\geq} \left(1 - \frac{y}{K}\right)$$

Let

$$x = 1 - \frac{y}{K}.$$

Then

$$\frac{K}{y} = \frac{1}{1-x}.$$

Substitute these two relationships into the inequality.

$$\ln \frac{1}{1-x} \stackrel{?}{\geq} x$$

Exponentiate both sides.

$$\frac{1}{1-x} \stackrel{?}{\geq} e^x$$

Substitute the Taylor series expansion for each expression.

$$1 + x + x^2 + x^3 + \cdots \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Therefore, dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation for $0 < y \leq K$ (corresponding to $0 \leq x < 1$).