# Problem 17

(a) Solve the Gompertz equation
$$dy/dt = ry\ln(K/y),$$
subject to the initial condition $y(0) = y_0$.
*Hint:* You may wish to let $u = \ln(y/K)$.

(b) For the data given in Example 1 in the text ($r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$), use the Gompertz model to find the predicted value of $y(2)$.

(c) For the same data as in part (b), use the Gompertz model to find the time $\tau$ at which $y(\tau) = 0.75K$.

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**Solution**

**Part (a)**

$$\frac{dy}{dt} = ry\ln\frac{K}{y}$$

Make the recommended substitution,

$$u = \ln\frac{y}{K} = -\ln\frac{K}{y}.$$

Then
$$y = Ke^u.$$

Our task now is to find what $dy/dt$ is in terms of this new variable. Differentiate both sides of this equation with respect to $t$, using the chain rule on the right side.

$$\frac{dy}{dt} = Ke^u\frac{du}{dt}$$

Substitute the previous three equations into the Gompertz equation to get an ODE for $u$.

$$Ke^u\frac{du}{dt} = r(Ke^u)(-u)$$

Divide both sides by $Ke^u$.
$$\frac{du}{dt} = -ru$$

Separate variables.
$$\frac{du}{u} = -r\,dt$$

Integrate both sides.
$$\ln|u| = -rt + C$$

Exponentiate both sides.

$$\begin{aligned}|u| &= e^{-rt+C}\\ &= e^C e^{-rt}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$u(t) = \pm e^C e^{-rt}$$

Use a new constant $A$ for $\pm e^C$.

$$u(t) = Ae^{-rt}$$

Now that $u$ is solved for, replace it with $\ln(y/K)$.

$$\ln \frac{y}{K} = Ae^{-rt}$$

Apply the initial condition $y(0) = y_0$ here to determine $A$.

$$\ln \frac{y_0}{K} = A$$

The general solution becomes

$$\ln \frac{y}{K} = \left(\ln \frac{y_0}{K}\right) e^{-rt}$$

$$\frac{y}{K} = \exp\left[\left(\ln \frac{y_0}{K}\right) e^{-rt}\right].$$

Therefore,

$$y(t) = K \exp\left[\left(\ln \frac{y_0}{K}\right) e^{-rt}\right].$$

## Part (b)

Plug in the numbers, $r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$, and $t = 2$ years.

$$y(2) = (80.5 \times 10^6 \text{ kg}) \exp\left[(\ln 0.25)e^{-(0.71)(2)}\right] \approx 5.8 \times 10^7 \text{ kg}$$

## Part (c)

Plug in the numbers, $r = 0.71$ per year, $y_0/K = 0.25$, and $y(t) = 0.75K$, and solve for $t = \tau$.

$$0.75K = K \exp\left[(\ln 0.25)e^{-(0.71)\tau}\right]$$

$$\exp\left[(\ln 0.25)e^{-(0.71)\tau}\right] = 0.75$$

$$(\ln 0.25)e^{-(0.71)\tau} = \ln 0.75$$

$$e^{-(0.71)\tau} = \frac{\ln 0.75}{\ln 0.25}$$

$$-0.71\tau = \ln \frac{\ln 0.75}{\ln 0.25}$$

Therefore,

$$\tau = \frac{1}{0.71} \ln \frac{\ln 0.25}{\ln 0.75} \approx 2.2 \text{ years}.$$