Problem 19

Consider a cylindrical water tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole of area a in the bottom of the tank. From Torricelli's principle in hydrodynamics (see Problem 6 in Section 2.3) it follows that the rate at which water flows through the hole is $\alpha a\sqrt{2gh}$, where h is the current depth of water in the tank, g is the acceleration due to gravity, and α is a contraction coefficient that satisfies $0.5 \le \alpha \le 1.0$.

(a) Show that the depth of water in the tank at any time satisfies the equation

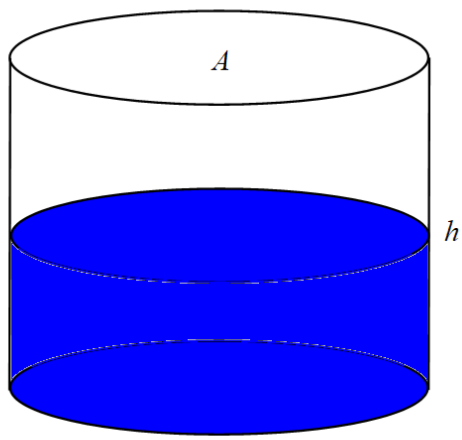
$$dh/dt = (k - \alpha a\sqrt{2gh})/A.$$

(b) Determine the equilibrium depth h_e of water, and show that it is asymptotically stable. Observe that h_e does not depend on A.

Solution

Part (a)

Start by drawing a schematic of the tank.



According to the law of conservation of mass, matter is neither created nor destroyed. If water flows into the tank at some rate, then it must leave at that same rate or else it will accumulate.

$$\text{rate of accumulation} = \text{rate in} - \text{rate out}$$

If m denotes the mass of the water, then dm/dt represents the rate that it accumulates. k is the volume of water per unit time that flows into the tank, so multiply it by the density ρ to get a mass flow rate. The rate that volume is lost through the hole is $\alpha a\sqrt{2gh}$. Multiply this by ρ to get a mass flow rate.

$$\frac{dm}{dt} = \rho k - \rho\alpha a\sqrt{2gh}$$

Mass is density times volume.

$$\frac{d(\rho V)}{dt} = \rho k - \rho\alpha a\sqrt{2gh}$$

$$\rho\frac{dV}{dt} = \rho k - \rho\alpha a\sqrt{2gh}$$

Divide both sides by ρ.

$$\frac{dV}{dt} = k - \alpha a\sqrt{2gh}$$

Volume is area times height.

$$\frac{d(Ah)}{dt} = k - \alpha a\sqrt{2gh}$$

$$A\frac{dh}{dt} = k - \alpha a\sqrt{2gh}$$

Therefore, the governing equation for the height of water in the tank is

$$\frac{dh}{dt} = \frac{k - \alpha a\sqrt{2gh}}{A}.$$

Part (b)

If equilibrium occurs, then there will be no accumulation, and the height will remain the same. Set $dh/dt = 0$ and solve for $h = h_e$.

$$0 = \frac{k - \alpha a\sqrt{2gh_e}}{A}$$

$$0 = k - \alpha a\sqrt{2gh_e}$$

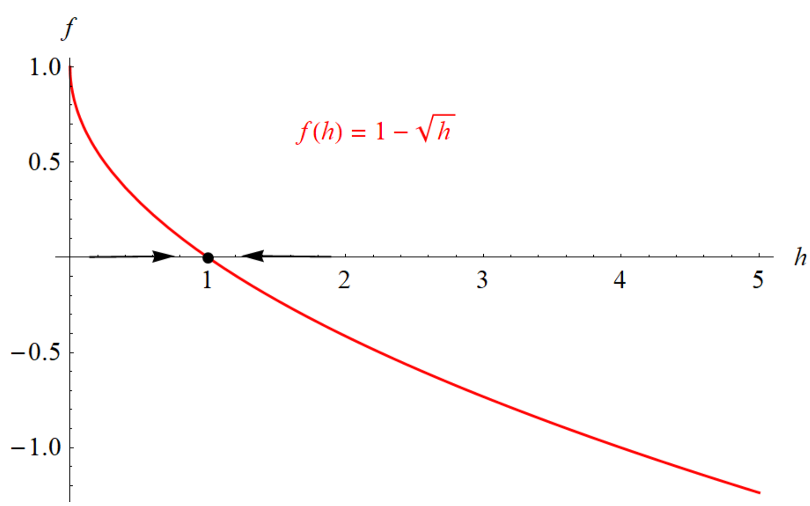
$$\alpha a\sqrt{2gh_e} = k$$

$$\sqrt{2gh_e} = \frac{k}{\alpha a}$$

$$2gh_e = \frac{k^2}{\alpha^2 a^2}$$

Therefore, the height at equilibrium is

$$h_e = \frac{k^2}{2g\alpha^2 a^2}.$$



Plotting $f(h) = (k - \alpha a\sqrt{2gh})/A$ versus h with the coefficients set to 1, we see that this equilibrium is stable.