

Problem 21

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

Although it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. Problems 20 and 21 explore some of the questions involved in formulating a rational strategy for managing the fishery.¹⁵

In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$dy/dt = r(1 - y/K)y - h. \quad (i)$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

- If $h < rK/4$, show that Eq. (i) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.
- Show that y_1 is unstable and y_2 is asymptotically stable.
- From a plot of $f(y)$ versus y , show that if the initial population $y_0 > y_1$, then $y \rightarrow y_2$ as $t \rightarrow \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.
- If $h > rK/4$, show that y decreases to zero as t increases, regardless of the value of y_0 .
- If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 7). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 20(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

Solution

Part (a)

The equilibrium points are found by setting $dy/dt = 0$ and solving the resulting equation for y .

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y - h = 0$$

$$ry - \frac{r}{K}y^2 - h = 0$$

$$y^2 - Ky + \frac{hK}{r} = 0$$

¹⁵An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

Use the quadratic equation to determine y .

$$y = \frac{K \pm \sqrt{K^2 - \frac{4Kh}{r}}}{2}$$

Therefore,

$$y_1 = \frac{K - \sqrt{K^2 - \frac{4Kh}{r}}}{2}$$

$$y_2 = \frac{K + \sqrt{K^2 - \frac{4Kh}{r}}}{2}.$$

These are equilibrium populations as long as

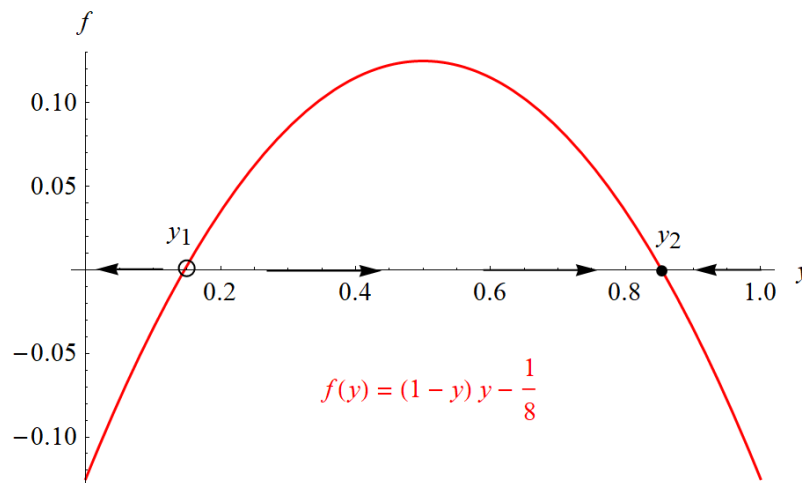
$$K^2 - \frac{4Kh}{r} > 0$$

$$\frac{4Kh}{r} < K^2$$

$$h < \frac{rK}{4}.$$

Part (b)

Plotting $f(y) = r(1 - y/K)y - h$ versus y with $r = 1$, $K = 1$, and $h = 1/8$, we see that the equilibrium at $y = y_1$ is unstable and the equilibrium at $y = y_2$ is stable.

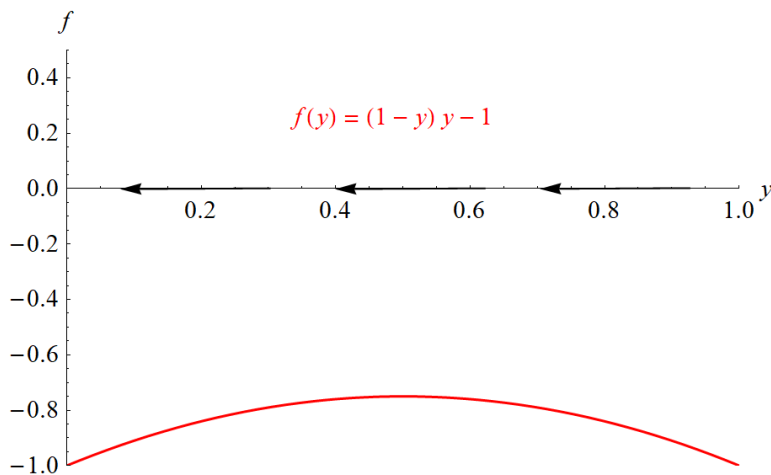


Part (c)

Suppose that the initial condition is $y(0) = y_0$. As indicated by the arrows above, if $y_0 > y_2$, then the population will tend to $y(t) = y_2$ as $t \rightarrow \infty$. If $y_0 = y_1$, then the population will remain there. If $y_0 < y_1$, then the population will tend to zero in a finite time because there are no equilibrium points to the left of $y = y_1$.

Part (d)

Suppose that $h > rK/4$, for example, using $r = 1$, $K = 1$, and $h = 1$. Then the entire graph of $f(y) = r(1 - y/K)y - h$ versus y lies below the y -axis, which means all the arrows point to the left.



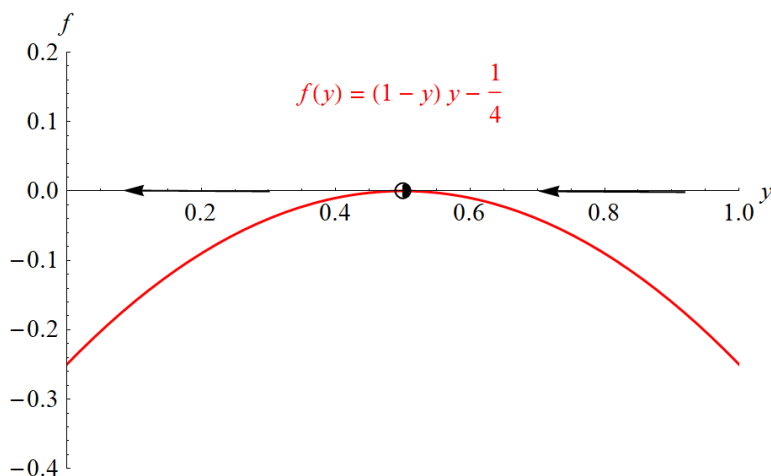
The population tends to zero regardless of what y_0 is.

Part (e)

Suppose that $h = rK/4$, for example, using $r = 1$, $K = 1$, and $h = 1/4$. Then there's only one equilibrium point at

$$y = \frac{K \pm \sqrt{K^2 - \frac{4Kh}{r}}}{2} = \frac{K}{2}.$$

The plot of $f(y) = r(1 - y/K)y - h$ versus y looks like this.



Even though the graph lies below the y -axis as in part (d), the difference is that there's now an equilibrium point at $y = K/2$. If $y_0 > K/2$, then the population will tend to $y(t) = K/2$ as $t \rightarrow \infty$. If $y_0 = K/2$, then the population will remain there. If $y_0 < K/2$, then the population will tend to zero in a finite time because there are no equilibria to the left of $y = K/2$. Because it's stable from one side and unstable from the other, $y = K/2$ is said to be semistable.