Problem 22

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[16] Problems 22 through 24 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$dy/dt = \alpha y(1 - y), \qquad y(0) = y_0, \tag{i}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

(a) Find the equilibrium points for the differential equation (i) and determine whether each is asymptotically stable, semistable, or unstable.

(b) Solve the initial value problem (i) and verify that the conclusions you reached in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

Solution

Part (a)

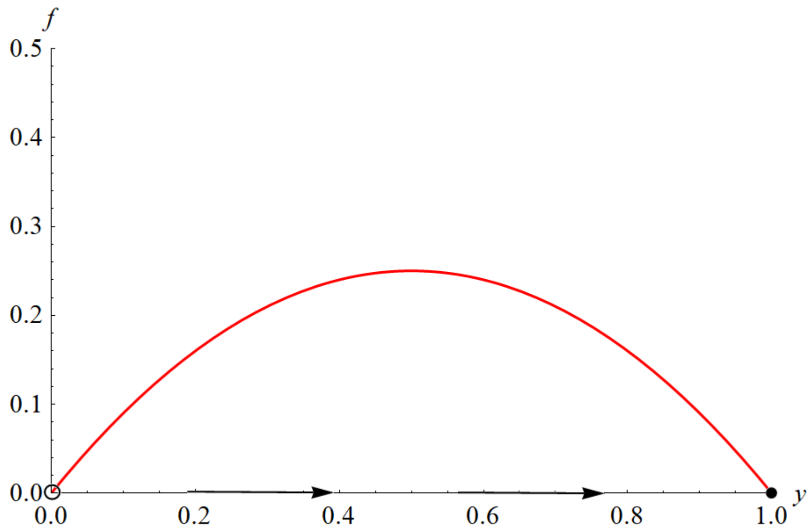
The equilibrium points are found by setting $dy/dt = 0$ and solving the resulting equation for y.

$$\alpha y(1 - y) = 0$$

$$y = 0 \quad \text{or} \quad y = 1$$

[16] A standard source is the book by Bailey listed in the references. The models in Problems 22, 23, and 24 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

To determine the nature of the equilibria, set $\alpha = 1$ and plot $f(y) = \alpha y(1-y)$ versus y.



Because the graph lies above the y-axis, the arrows point to the right, making $y = 0$ an unstable equilibrium solution and $y = 1$ a stable equilibrium solution. What this means is that even if only one individual is infected in a population of one billion, the entire population will eventually be infected.

Part (b)

$$\frac{dy}{dt} = \alpha y(1-y)$$

Solve the ODE by separating variables.

$$\frac{dy}{y(1-y)} = \alpha\, dt$$

Integrate both sides.

$$\int \frac{dy}{y(1-y)} = \alpha t + C$$

Use partial fraction decomposition to split up the integral on the left side.

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \alpha t + C$$

$$\int \left(\frac{1}{y} - \frac{1}{y-1}\right) dy = \alpha t + C$$

$$\ln|y| - \ln|y-1| = \alpha t + C$$

$$\ln\left|\frac{y}{y-1}\right| = \alpha t + C$$

Apply the initial condition $y(0) = y_0$ here to determine C.

$$\ln\left|\frac{y_0}{y_0-1}\right| = C$$

As a result, the previous equation becomes

$$\ln\left|\frac{y}{y-1}\right| = \alpha t + \ln\left|\frac{y_0}{y_0-1}\right|.$$

Since y and y_0 are between 0 and 1, the absolute value signs can be removed by writing the denominators as $1-y$ and $1-y_0$.

$$\ln\frac{y}{1-y} = \alpha t + \ln\frac{y_0}{1-y_0}$$

Exponentiate both sides.

$$\begin{aligned}\frac{y}{1-y} &= \exp\left(\alpha t + \ln\frac{y_0}{1-y_0}\right)\\ &= \exp(\alpha t)\exp\left(\ln\frac{y_0}{1-y_0}\right)\\ &= e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)\end{aligned}$$

Multiply both sides by $1-y$.

$$y = e^{\alpha t}\left(\frac{y_0}{1-y_0}\right) - e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)y$$

Solve for y.

$$y\left[1 + e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)\right] = e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)$$

Therefore,

$$\begin{aligned}y(t) &= \frac{e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)}{1 + e^{\alpha t}\left(\frac{y_0}{1-y_0}\right)}\\ &= \frac{e^{\alpha t}(y_0)}{(1-y_0) + e^{\alpha t}(y_0)}\\ &= \frac{y_0}{(1-y_0)e^{-\alpha t} + y_0}.\end{aligned}$$

If there are no infected individuals initially ($y_0 = 0$), then $y(t) = 0$. However, if $0 < y_0 < 1$, then the exponential term decays to zero as $t \to \infty$, leaving $y(t) = y_0/y_0 = 1$. If everyone is infected initially ($y_0 = 1$), then $y(t) = y_0/y_0 = 1$.