Problem 23

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[16] Problems 22 through 24 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

Some diseases (such as typhoid fever) are spread largely by carriers, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \tag{i}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = -\alpha xy. \tag{ii}$$

(a) Determine y at any time t by solving Eq. (i) subject to the initial condition $y(0) = y_0$.

(b) Use the result of part (a) to find x at any time t by solving Eq. (ii) subject to the initial condition $x(0) = x_0$.

(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

Solution

Part (a)

Bring βy to the left side in Eq. (i).

$$\frac{dy}{dt} + \beta y = 0$$

Solve the ODE by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \beta\, ds\right) = e^{\beta t}$$

Proceed with the multiplication.

$$e^{\beta t}\frac{dy}{dt} + \beta e^{\beta t} y = 0$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^{\beta t} y) = 0$$

Integrate both sides with respect to t.

$$e^{\beta t} y = A$$

[16] A standard source is the book by Bailey listed in the references. The models in Problems 22, 23, and 24 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

Divide both sides $e^{\beta t}$ to solve for y.

$$y(t) = Ae^{-\beta t}$$

Apply the initial condition $y(0) = y_0$ to determine A.

$$y(0) = A = y_0$$

Therefore,

$$y(t) = y_0 e^{-\beta t}.$$

Part (b)

Substitute the result for y into Eq. (ii).

$$\frac{dx}{dt} = -\alpha x(y_0 e^{-\beta t})$$

Solve this ODE by separating variables.

$$\frac{dx}{x} = -\alpha y_0 e^{-\beta t}\, dt$$

Integrate both sides.

$$\ln|x| = \frac{\alpha}{\beta} y_0 e^{-\beta t} + C$$

Exponentiate both sides.

$$\begin{aligned} |x| &= \exp\left(\frac{\alpha}{\beta} y_0 e^{-\beta t} + C\right) \\ &= e^C \exp\left(\frac{\alpha}{\beta} y_0 e^{-\beta t}\right) \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$x(t) = \pm e^C \exp\left(\frac{\alpha}{\beta} y_0 e^{-\beta t}\right)$$

Use a new constant B for $\pm e^C$.

$$x(t) = B \exp\left(\frac{\alpha}{\beta} y_0 e^{-\beta t}\right)$$

Now apply the initial condition $x(0) = x_0$ to determine B.

$$x(0) = B \exp\left(\frac{\alpha}{\beta} y_0\right) = x_0 \quad \to \quad B = x_0 \exp\left(-\frac{\alpha}{\beta} y_0\right)$$

The previous equation then becomes

$$\begin{aligned} x(t) &= x_0 \exp\left(-\frac{\alpha}{\beta} y_0\right) \exp\left(\frac{\alpha}{\beta} y_0 e^{-\beta t}\right) \\ &= x_0 \exp\left(-\frac{\alpha}{\beta} y_0 + \frac{\alpha}{\beta} y_0 e^{-\beta t}\right). \end{aligned}$$

Therefore,

$$x(t) = x_0 \exp\left[-\frac{\alpha}{\beta} y_0 (1 - e^{-\beta t})\right] \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = x_0 \exp\left[-\frac{\alpha}{\beta} y_0 (1 - 0)\right] = x_0 \exp\left(-\frac{\alpha}{\beta} y_0\right).$$