

Problem 1

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$(2x + 3) + (2y - 2)y' = 0$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(2x + 3) = \frac{\partial}{\partial x}(2y - 2) = 0.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 2x + 3 \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 2y - 2. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ .

$$\psi(x, y) = x^2 + 3x + f(y)$$

Here f is an arbitrary function of y . Differentiate both sides with respect to y .

$$\psi_y(x, y) = f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 2y - 2 \quad \rightarrow \quad f(y) = y^2 - 2y.$$

As a result, a potential function is

$$\psi(x, y) = x^2 + 3x + y^2 - 2y.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

Dividing both sides by dx , we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

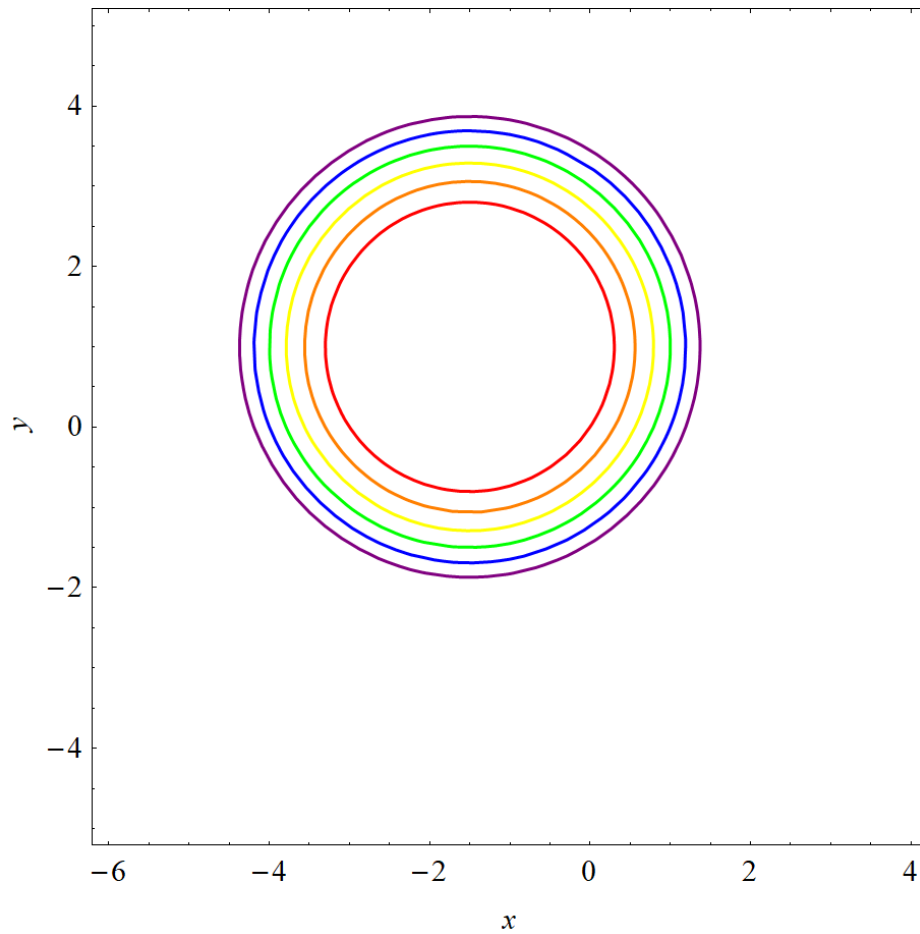
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x .

$$\psi(x, y) = C$$

Therefore,

$$x^2 + 3x + y^2 - 2y = C.$$



The general solution is a family of circles. This figure illustrates several of them. In red, orange, yellow, green, blue, and purple are $C = 0$, $C = 1$, $C = 2$, $C = 3$, $C = 4$, and $C = 5$, respectively.