Problem 3

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(3x^2 - 2xy + 2) = \frac{\partial}{\partial x}(6y^2 - x^2 + 3) = -2x.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2 \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 6y^2 - x^2 + 3. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = x^3 - x^2 y + 2x + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = -x^2 + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 6y^2 + 3 \quad \rightarrow \quad f(y) = 2y^3 + 3y.$$

As a result, a potential function is

$$\psi(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

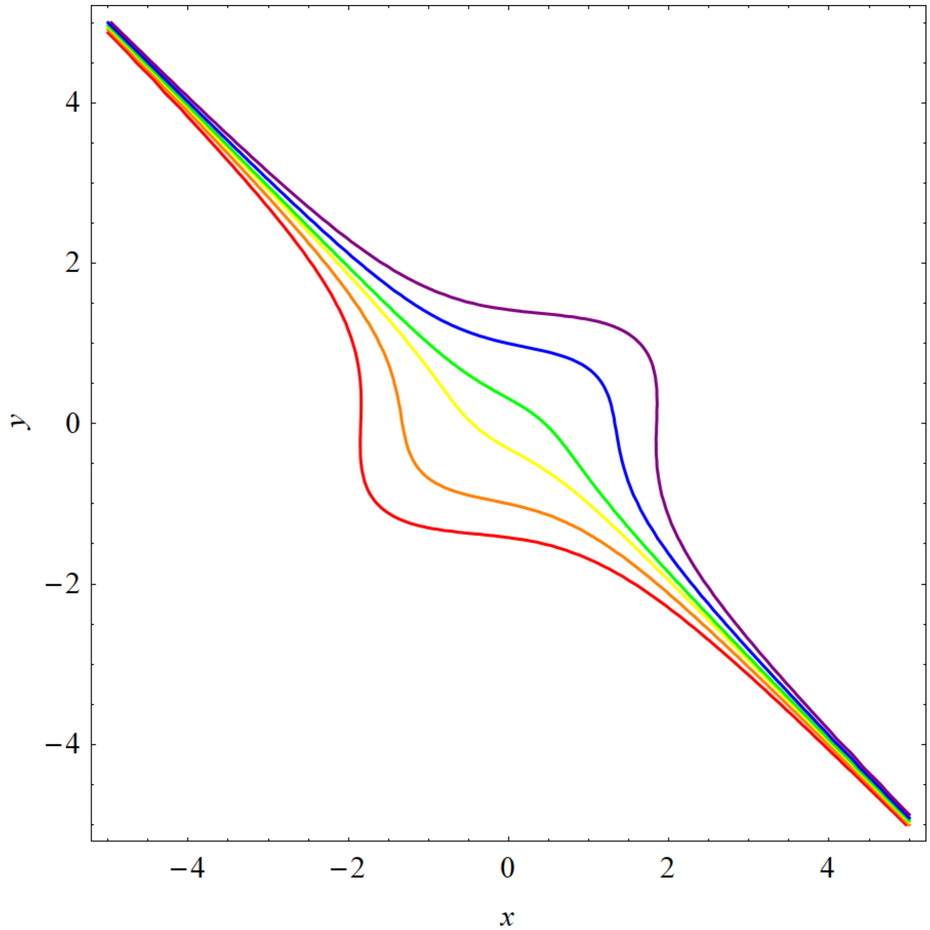
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$x^3 - x^2 y + 2x + 2y^3 + 3y = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.