Problem 12

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$\frac{x}{(x^2+y^2)^{3/2}}+\frac{y}{(x^2+y^2)^{3/2}}\frac{dy}{dx}=0$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right)=\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right)=-\frac{3xy}{(x^2+y^2)^{5/2}}.$$

That means there exists a potential function $\psi=\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x}=\frac{x}{(x^2+y^2)^{3/2}} \tag{1}$$

$$\frac{\partial\psi}{\partial y}=\frac{y}{(x^2+y^2)^{3/2}}. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x,y)=-\frac{1}{\sqrt{x^2+y^2}}+f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y)=\frac{y}{(x^2+y^2)^{3/2}}+f'(y)$$

Comparing this to equation (2), we see that

$$f'(y)=0 \quad\to\quad f(y)=0.$$

As a result, a potential function is

$$\psi(x,y)=-\frac{1}{\sqrt{x^2+y^2}}.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=0. \tag{3}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi=\frac{\partial\psi}{\partial x}\,dx+\frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx}=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

$$-\frac{1}{\sqrt{x^2 + y^2}} = C$$

Square both sides.

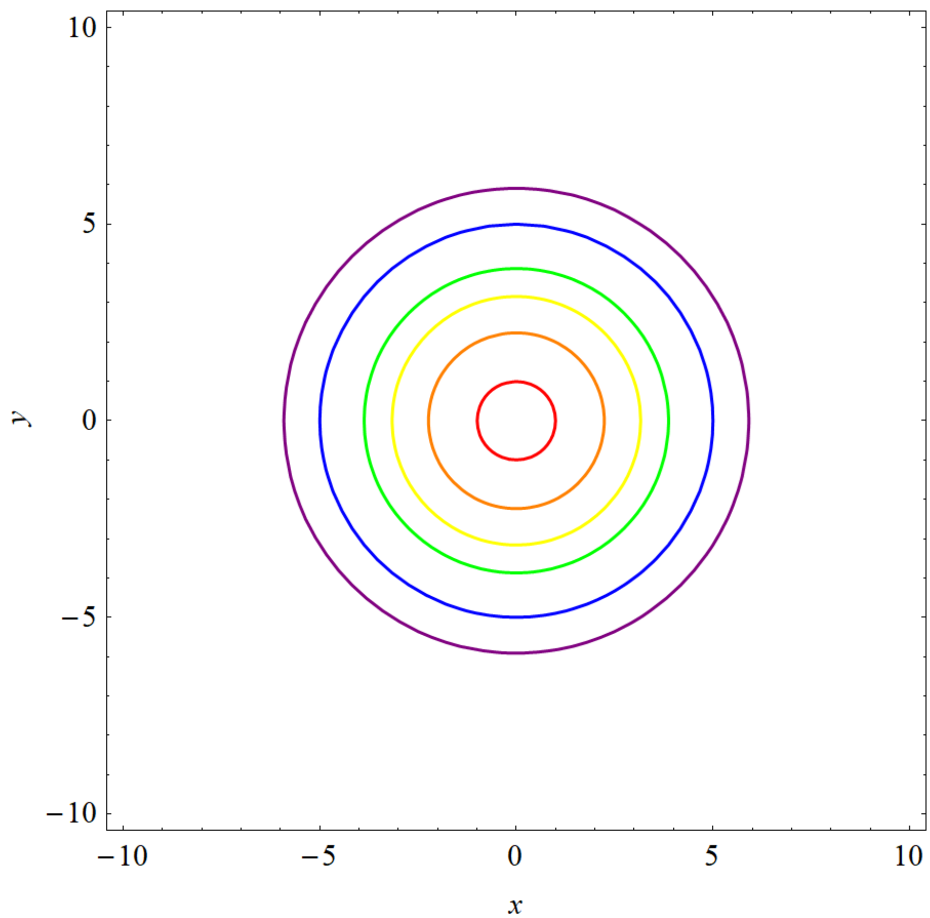
$$\frac{1}{x^2 + y^2} = C^2$$

Invert both sides.

$$x^2 + y^2 = \frac{1}{C^2}$$

Use a new constant for the right side.

$$x^2 + y^2 = k$$



The general solution is a family of circles. This figure illustrates several of them. In red, orange, yellow, green, blue, and purple are $k = 1$, $k = 5$, $k = 10$, $k = 15$, $k = 25$, and $k = 35$, respectively.